Analytical approach to solve Numerical and Transcendental Equations Using Numerical Methods

[1]Rovin Kumar, [2]Ravindra Kumar Upadhyay
Email: drrovinkr9393@gmail.com, rkravin10@gmail.com
Assistant Professor
IIMT College of Management, Greater Noida, India

Abstract:

We know An expression of the form $(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$, $a_0 \neq 0$ is called a polynomial of degree 'n'and the polynomial $f(x) = 0$ is called an algebraic equation of n^{th} degree. If (x) contains trigonometric, logarithmic or exponential functions, then $f(x) = 0$ is called a transcendental equation. For example $x^2 + 2\sin x + e^x = 0$ is a transcendental equation.

If $(x)$is an algebraic polynomial of degree less than or equal to 4, direct methods for finding the roots of such equation are available. But if $(x)$is of higher degree or it involves transcendental functions, direct methods do not exist and we need to apply numerical methods to find the roots of the equation $f(x) = 0$.

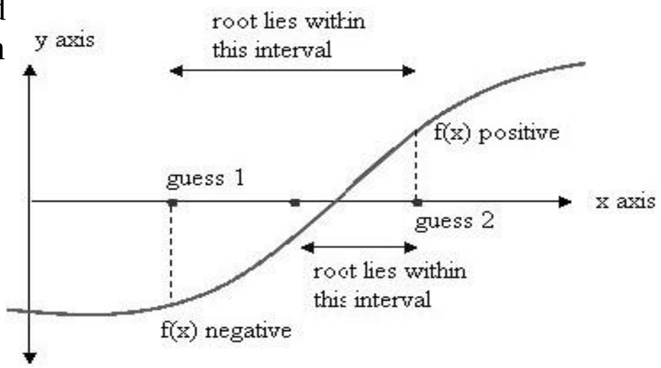
Keyword: Algebraic Equation, Transcedental Equations, Newton Raphson Method, Regula falsi Method, Bisection Methods

Introduction:

If α is root of the equation $(x) = 0$, then $f(\alpha) = 0$
Every equation of n^{th} degree has exactly n roots (real or imaginary) If $(x)$is a continuous function in a closed interval $[a, b]$ and $f(a)$& $f(b)$ are having opposite signs, then the equation
$(x) = 0$ has at least one real root or odd number of roots between $a$and b.
If $f(x)$is a continuous function in the closed interval $[a, b]$and $f(a)$ & $f(b)$ are of same signs, then the equation $f(x) = 0$ has no root or even number of roots between $a$and b.

Order of convergence: For any iterative numerical method, each successive iteration gives an approximation that moves progressively closer to actual solution. This is known as convergence. Any numerical method is said have order of convergence ρ, if ρ is the largest positive number such that $|\epsilon_{n+1}| \leq k|\epsilon_n|$, where ϵ_n and ϵ_{n+1} are errors in n^{th} and $(n+1)^{th}$ iterations, k is a finite positive constant.

- **Bisection Method (or Bolzano Method)**Bisection method is used to find an approximate root in an interval by repeatedly bisecting into subintervals. It is a very simple and robust method but it is also relatively slow. Because of this it is often used to obtain a rough approximation to a solution which is then used as a starting point for more rapidly converging methods. This method is based on the intermediate value theorem for continuous functions.



Algorithm:

Let (x) be a continuous function in the interval $[a, b]$, such that $f(a)$and $f(b)$ are of opposite signs, i.e. $f(a).(b) < 0$.

Step1. Take the initial approximation given by $x_0 = \frac{a+b}{2}$, one of the three conditions arises for finding the 1st approximation x_1

i. $(x_0) = 0$, we have a root at x_0.

ii. If $(a).f(x_0) < 0$, the root lies between a and x_0 $\therefore x_1 = \frac{a+x_0}{2}$ and repeat the procedure by halving the interval again.

iii. I$f\ (b).(x_0) < 0$, the root lies between x_0 and b $\therefore x_1 = \frac{x_0+b}{2}$ and repeat the procedure by halving the interval again.

iv. Continue the process until root is found to be of desired accuracy.

Remarks:

- Convergence is not unidirectional as none of the end points is fixed. As a result convergence of Bisection method is very slow.
- Repeating the procedure n times, the new interval will be exactly half the length of the previous one, until the root is found of desired accuracy (error less than ∈). ∴ and at the end ofn^{th} iteration, the interval containing the root will be of length
-

$\frac{|b-a|}{2^n}$, such that $\frac{|b-a|}{2^n} < \in$

$$\Rightarrow \log \frac{|b-a|}{2^n} < \log \in$$

$$\Rightarrow \log|b-a| - \log 2^n < \log \in$$

$$\Rightarrow \log|b-a| - \log \in < n \log 2$$

$$\Rightarrow n > \frac{\log|b-a| - \log \in}{\log 2}$$

∴In bisection method, the minimum number of iterations required to achieve the desired accuracy (error less than ∈)are $\frac{\log\frac{|b-a|}{\epsilon}}{\log 2}$.

Example1: Apply bisection method to find a root of the equation $x^4 + 2x^3 - x - 1 = 0$

Solution:$(x) = x^4 + 2x^3 - x - 1$

Here $(0) = -1$and $f(1) = 1 \Rightarrow f(0).(1) < 0$

Also $(x)$is continuous in $[0,1]$, ∴ atleast one root exists in $[0,1]$

Initial approximation: $a = 0, b = 1$

$x_0 = \frac{0+1}{2} = .5,\ (0.5) = -1.1875,\ f(0.5).(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = \frac{0.5+1}{2} = 0.75,\ (0.75) = -0.5898,\ f(0.75).(1) < 0$

Second approximation: $a = 0.75, b = 1$

$x_2 = \frac{0.75+1}{2} = 0.875,\ (0.875) = 0.051,\ f(0.75).(0.875) < 0$

Third approximation: $a = 0.75, b = 0.875$

$x_3 = \frac{0.75+0.875}{2} = 0.8125,\ (0.8125) = -0.30394, f(0.8125).(0.875) < 0$

Fourth approximation: $a = 0.8125, b = 0.875$

$x_4 = \frac{0.8125+0.875}{2} = 0.84375,(0.84375) = -0.135, f(0.84375).\ (0.875) < 0$

Fifth approximation: $a = 0.84375, b = 0.875$

$x_5 = \frac{0.84375+0.875}{2} = 0.8594,\ (0.8594) = -0.0445, .(0.8594).(0.875) < 0$

Sixth approximation: $a = 0.8594, b = 0.875$

$x_6 = \frac{0.8594 + 0.875}{2} = 0.8672, (0.8672) = 0.0027,\ f(0.8594).(0.8672). < 0$

Seventh approximation: $a = 0.8594, b = 0.8672$

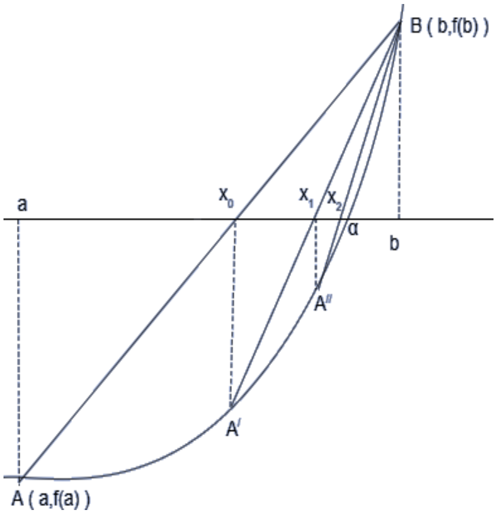
$x_7 = \frac{0.8594+0.8672}{2} = 0.8633$

First 2 decimal places have been stabilized; hence 0.8633 is the real root correct to twodecimal places.

Hence 2.5952 is the real root correct to three decimal places.

Regula- Falsi Method (Geometrical Interpretation)

Regula-Falsi method is also known as method of false position as false position of curve is taken as initial approximation. Let $y = (x)$ be represented by the curve AB.The real root of equation $f(x) = 0$ is α as shown in adjoining figure. The false position of curve AB is taken as chord AB and initial approximation x_0 is the point of intersection of chord AB with x-axis. Successive approximations x_1, x_2, ...are given by point of intersection of chord $A'B$, $A''B$, ...with $x -$ axis, until the root is found to be of desired accuracy.



Now equation of chord AB in two-point form is given by:

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

To find x_0 (point of intersection of chord AB with x-axis), put $y = 0$

$$\Rightarrow -f(a) = \frac{f(b)-f(a)}{b-a}(x_0 - a)$$

$$\Rightarrow (x_0 - a) = \frac{-(b-a)f(a)}{f(b)-f(a)}$$

$$\Rightarrow x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$$

Repeat the procedure until the root is found to the desired accuracy.

Example2: Apply Regula-Falsi method to find a root of the equation $x^3 + x - 1 = 0$ correct to two decimal places.

Solution:$(x) = x^3 + x - 1$

Here $(0) = -1$ and $f(1) = 1 \Rightarrow f(0).(1) < 0$

Also $(x)$is continuous in [0,1], ∴ atleast one root exists in [0,1]

Initial approximation:$_0 = a - \frac{(b-a)}{f(b)-f(a)}(a)$; $a = 0$, $b = 1$

$$\Rightarrow x_0 = 0 - \frac{(1-0)}{(1)-f(0)}(0) = 0 - \frac{1}{1-(-1)}(-1) = 0.5$$

$(0.5) = -0.375,\ f(0.5).(1) < 0$

First approximation: $a = 0.5$, $b = 1$

$$x_1 = 0.5 - \frac{(1-0.5)}{(1)-f(0.5)}(0.5) = 0 - \frac{0.5}{1-(-0.375)}(-0.375) = 0.636$$

$(0.636) = -0.107,\ f(0.636).(1) < 0$

Second approximation: $a = 0.636$, $b = 1$

$$x_2 = 0.636 - \frac{(1-0.636)}{(1)-f(0.636)}(0.636) = 0.636 - \frac{0.364}{1-(-0.107)}(-0.107) = 0.6711$$

https://doi.org/10.5281/zenodo.13268456

$(0.6711) = -0.0267,\ f(0.6711).(1) < 0$

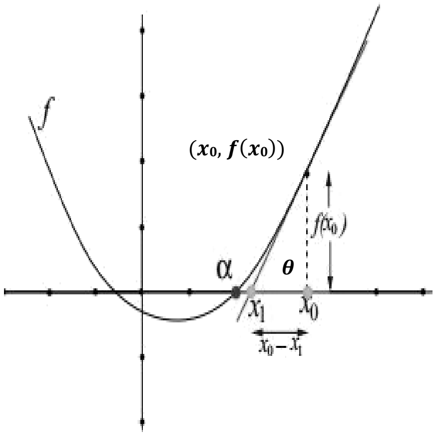
Third approximation: $a = 0.6711, b = 1$

$$x_3 = .6711 - \frac{(1-0.6711)}{(1)-f(0.6711)}(.6711) = .6711 - \frac{0.3289}{1-(-0.0267)}(-.0267) = 0.6796$$

First 2 decimal places have been stabilized; hence 0.6796 is the real root correct to two decimal places.

Newton-Raphson Method (Geometrical Interpretation)

Newton-Raphson method named after Isaac Newton and Joseph Raphson is a powerful technique for solving equations numerically. The Newton-Raphson method in one variable is implemented as follows:



Let α be an exact root and x_0 be the initial approximate root of the equation $(x) = 0$. First approximation x_1 is taken by drawing a tangent to curve $y = (x)$ at the point $(x_0, f(x_0))$. If θ is the angle which tangent through the point $(x_0, (x_0))$ makes with x- axis, thenslope of the tangent is given by:

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \frac{(x_1)}{f'(x_1)}$

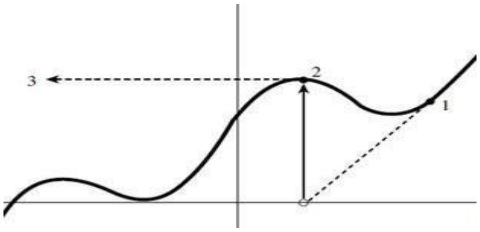
$$\vdots$$

The required root to desired accuracy is obtained by drawing tangents to the curve at points $(x, f(x_n))$ successively.

$$\therefore\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

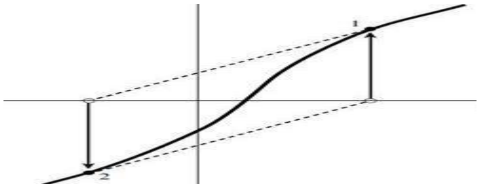
Newton-Raphson method works very fast but sometimes it fails to converge as shown below:

Case I:



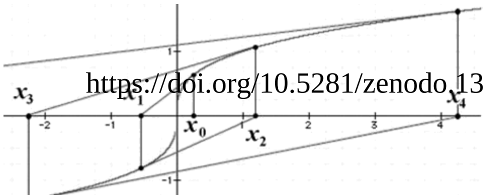
If any of the approximations encounters a zero derivative (extreme point), then the tangent at that point goes parallel to x-axis, resulting in no further approximations as shown in given figure where third approximation tends to infinity.

Case II:



Sometimes Newton-Raphson method may run into an infinite cycle or loop as shown in adjoining figure. Change in initial approximation may untangle the problem.

Case III:



In case of a point of discontinuity, as shown in given figure, subsequent roots may diverge instead of converging.

Remarks:

- Newton-Raphson method can be used for solving both algebraic and transcendental equations and it can also be used when roots are complex.
- Initial approximation x_0 can be taken randomly in the interval $[a, b]$, such that $f(a).f(b) < 0$
- Newton-Raphson method has quadratic convergence, but in case of bad choice of x_0 (the initial guess), Newton- Raphson method may fail to converge
- This method is useful in case of large value of $f'(x_n)$ i.e. when graph of $f(x)$ while crossing x -axis is nearly vertical.

Example 3: Find the approximate value of $\sqrt{28}$ correct to 3 decimal places using Newton Raphson method.

Solution: $x = \sqrt{28} \quad \Rightarrow x^2 - 28 = 0$

$\therefore (x) = x^2 - 28$

$\Rightarrow f'(x) = 2x$

Here $(5) = -3$ and $f(6) = 8 \Rightarrow f(5).(6) < 0$

Also (x) is continuous in [5,6], ∴ atleast one root exists in [5,6]

Initial approximation: Let initial approximation (x_0) in the interval [5,6] be 5.5

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{(x_0)}{f'(x_0)}$, where $x_0 = 5.5,\ (5.5) = 2.25, f'(5.5) = 11$

$\Rightarrow x_1 = 5.5 - \frac{\mathbf{2.25}}{11} = 5.2955$

Second approximation:

$x_2 = x_1 - \frac{(x_1)}{f'(x_1)}$, where $x_1 = 5.2955, (5.2955) = 0.0423, f'(5.2955) = 10.591$

$\Rightarrow x_2 = 5.2955 - \frac{0.0423}{10.591} = 5.2915$

Third approximation:

$x_3 = x_2 - \frac{(x_2)}{f'(x_2)}$, where $x_2 = 5.2915, (5.2915) = -0.00003, f'(5.2915) = 10.583$

$\Rightarrow x_3 = 5.2915 - \frac{-0.00003}{10.583} = 5.2915$

https://doi.org/10.5281/zenodo.13268456

Hence value of $\sqrt{28}$ correct to three decimal places is 5.2915

Example 4: Use Newton-Raphson method to find a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places.

Solution: $(x) = x \sin x + \cos x$

$$\Rightarrow f'(x) = x \cos x + \sin x - \sin x = x \cos x$$

Here $f\left(\frac{\pi}{2}\right) = 1.5708$ and $(\pi) = -1 \Rightarrow f\left(\frac{\pi}{2}\right)(\pi) < 0$

Also (x) is continuous in $\left[\frac{\pi}{2}, \pi\right]$ ∴ atleast one root exists in $\left[\frac{\pi}{2}, \pi\right]$

Initial approximation: Let initial approximation (x_0) in the interval $\left[\frac{}{2}, \pi\right]$ be π

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{(x_0)}{f'(x_0)}$, where $x_0 = \pi$, $(\pi) = -1$, $f'(\pi) = -3.1416$

$$\Rightarrow x_1 = 3.1416 - \frac{-1}{-3.1416} = 2.8233$$

Second approximation:

$x_2 = x_1 - \frac{(x_1)}{f'(x_1)}$, where $x_1 = 2.8233$, $(2.8233) = -0.0662$, $f'(2.8233) = -2.6815$

$$\Rightarrow x_2 = 2.8233 - \frac{-0.0662}{-2.6815} = 2.7986$$

Third approximation:

$x_3 = x_2 - \frac{(x_2)}{f'(x_2)}$, where $x_2 = 2.798$, $(2.7986) = -0.0006$, $f'(2.7986) = -2.6356$

$$\Rightarrow x_3 = 2.7986 - \frac{-0.0006}{-2.6356} = 2.7984$$

Hence a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places is 2.7984

Generalized Newton's Method for Multiple Roots

Result: If α is a root of equation $(x) = 0$ with multiplicity m, then it is also a root of equation $f'(x) = 0$ with multiplicity $(m - 1)$ and also of the equation $f''(x) = 0$ with multiplicity $(m - 1)$ and so on.

For example $(x - 1)^3 = 0$ has '1' as a root with multiplicity 3

$3(x - 1)^2 = 0$ has '1' as the root with multiplicity 2

$6(x - 1) = 0$ has '1' as the root with multiplicity 1

∴ The expressions $x_n - m\frac{f(x_n)}{f'(x_n)}$, $x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$, $x_n - (m-2)\frac{f''(x_n)}{f'''(x_n)}$ are equivalent.

Generalized Newton's method is used to find repeated roots of an equation as is given as:

If α be a root of equation $(x) = 0$ which is repeated m times,

Then $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \sim x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$

Converegence of Newton Raphson Method

Let α be an exact root of the equation $(x) = 0$

$\Rightarrow (\alpha) = 0$

Also let x_n and x_{n+1} be two successive approximations to the root α.

If $\in_n$ and $\in_{n+1}$ are the corresponding errors in the approximations x_n and x_{n+1}

Then $x_n = \alpha + \in_n$ ....(1)

and $x_{n+1} = \alpha + \in_{n+1}$ ...(2)

Now by Newton Raphson method

$x_{n+1} = x_n - \frac{(x_n)}{f'(x_n)}$ ...(3)

Using (1) and (2) in (3)

$$\Rightarrow \alpha + \in_{n+1} = \alpha + \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$\Rightarrow \in_{n+1} = \frac{\in_n f'(\alpha + \in_n) - f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$\Rightarrow \in_{n+1} = \frac{\in_n [f'(\alpha) + \in_n f''(\alpha) + \ldots] - [f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots]}{f'(\alpha) + \in_n f''(\alpha) + \ldots}$$ By Taylor's expansion

$$\Rightarrow \in_{n+1} = \frac{\in_n^2 f''(\alpha) - \frac{\in_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha)\left[1 + \frac{\in_n f''(\alpha)}{f'(\alpha)} + \cdots\right]} \qquad \because (\alpha) = 0$$

$$\Rightarrow \in_{n+1} = \left[\frac{\in_n^2 f''(\alpha)}{2\ f'(\alpha)} + \cdots\right]\left[1 + \frac{\in_n f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$\Rightarrow \in_{n+1} = \left[\frac{\in_n^2 f''(\alpha)}{2\ f'(\alpha)} + \cdots\right]\left[1 - \frac{\in_n f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$\Rightarrow \in_{n+1} = \frac{\in_n^2 f''(\alpha)}{2\ f'\ \ 2\ f'(\alpha)}$$ Neglecting higher order terms

∴ Newton Raphson method has convergence of order 2 or quadratic convergence.

Conclusion:

The Newton-Raphson Method and the Regula Falsi (False Position) Method are both numerical techniques used to approximate the roots of a real-valued function. However, they differ in several ways, and the choice between them depends on the specific characteristics of the problem at hand.

Newton Raphson converges faster than the Regula Falsi method when the initial guess .The rate of convergence is usually quadratic, which means it can achieve a high level of accuracy in fewer iterations near a simple root.

Regula Falsi also an iterative method but relies on linear interpolation between two points to approximate the root. Convergence can be slower than the Newton-Raphson method, especially if the interval [a,b] containing the root is wide or if the function has complex behavior. The rate of convergence is linear, which means the number of iterations required for a certain level of accuracy may be higher.

In Newton Raphson Method requires an initial guess close to the root to ensure convergence. It May fail or converge to a different root if the initial guess is not well-chosen.

While in Regula Falsi Method Works with any interval [a,b] where f(a) and (f(b) have opposite signs. Doesn't require a particularly accurate initial guess but may converge more slowly if the interval is wide.

In summary, the choice between the Newton-Raphson Method and the Regula Falsi Method depends on the specific problem and the characteristics of the function you are trying to find roots for. If you have a good initial guess and can compute or approximate the derivative easily, the Newton-Raphson Method is often preferred for its faster convergence. However, if you need a more robust method that doesn't rely on a precise initial guess or the availability of derivatives, the Regula Falsi Method can be a better choice.

References:

[1] R. L. Burden and J. D. Faires, Numerical Analysis, Ninth Edition, Brooks/ Cole, Cengage Learning, 2011.

[2] Thinzar, and N. Aye,"Detection the storm movement by sub pixel registration approach of Newton-Raphson method " International Journal of e-Education, e-Business, e-Management and e-Learning, Vol. 4, No. 1, 2014

https://doi.org/10.5281/zenodo.13268456

[3] M.K. Jain, S.R.K. Iyengar, R.K. Jain, “Numerical Methods for Scientific and Engineering Computation”, 5th edition, New Age International Publishers.

[4] Acton, F. S. Ch. 2 in Numerical Methods That Work. Washington, DC: Math. Assoc. Amer., 1990.

[5] Ortega, J. M. and Rheinboldt, W. C. Iterative Solution of Nonlinear Equations in Several Variables. Philadelphia, PA: SIAM, 2000

[6] K.E. Atkinson. An Introduction to Numerical Analysis. Wiley, New York, second edition, 1989.

[7] A. Ralston and P. Rabinowitz. A First Course in Numerical Analysis. McGraw-Hill, New York, second edition, 1978.