# Mutually Singular implicit functions

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**Abstract** A continuously differentiable (or **smooth** or in $C'(X_1)$) function $\pi_1$from an open set $\Gamma$ which is a subset of a linear space $X_1$ of dimension $2n$ into a linear space $X_2$ of dimension $n$ has an inverse onto a subspace upto equal dimension. If $(s, t)$ is in $\Gamma$ where $\pi_1(s,t) = \mathbf{0}, \pi_1'(s,t) = M$invertible at $(s, t)$, then $\pi_1$is invertible in the neighbourhood of $(s, t)\, \epsilon \Gamma$. Upto bijection, the invertibility of a multi variate function $\pi_1$is possible and for this $\pi_1$ from $X_1$ is mapped onto $X_2$ so as to get invertible for the 1st $n$ components of the $2n$ – tuple. In the present discussion, we focus on two multi variate functions $\pi_1$ and $\pi_2$ or linear mappings from $X_1$ into $X_2$ such that both are smooth in two open sets $\Gamma_1$ and $\Gamma_2$ subsets of $X_1$ taking the 1st $n$ – components of $2n$ – tuples in $X_1$ onto $X_2$ while $\pi_2$ is from the open set $\Gamma_2$ of $X_1$ taking the 2nd set of $n$ – components of $2n$ – tuples in $X_1$ such that $\pi_1'$ is invertible at $(s_1, t_1)\epsilon\Gamma_1$ and $\pi_2{}'$ is invertible at $(s_2, t_2)\epsilon\Gamma_2$ . The sum of derivative of the implicit functions is a continuously differentiable function gives

$$\rho_1{}'(t) + \rho_2{}'(s) = - \left\{M_{1\xi}^{-1}M_{1ῄ} + M_{2ῄ}^{-1}M_{2\xi}\right\} \qquad \text{...... 0.1}$$

with $\rho_1$ and $\rho_2$ are the implicit functions, $M_1 = M_{1\xi} + M_{1ῄ}$ and $M_2 = M_{2\xi} + M_{2ῄ}$ in which $M_{1\xi}$ and $M_{2ῄ}$ are the invertible parts while $M_{1ῄ}$ and $M_{2\xi}$ are singular parts.
Further, the null space of $\pi_1$ is the range space of $\pi_2$ and vice versa leading to $\pi_1$ and $\pi_2$ are mutually singular, denoted by $\pi_{1\perp}\pi_2$ ......0.2

**Key words**: linear space, multi variate function or a vector valued function, nonsingular mapping, mutually singular mapping, invertible mapping, null space $\vartheta(\pi)$, range space $\gamma(\pi)$, dimension of a linear space, span, rank and nullity of a linear transformation, contraction, upto invertibility, subset, open mapping.

## Chapter 1:

**Introduction:** A linear space $X$ of dimension $n$ is of the form $\{(s_1, s_2, \dots, s_n): s_i, 1 \leq i \leq n\, \epsilon F\}$ where $F$ is a field. Each $n$ – tuple of $X$ can be written as a linear combination of the $n$ basis vectors of $X$. A proper subset of basis vectors can span $\{(t_1, t_2, \dots, t_m): t_i, 1 \leq i \leq m\, \epsilon F, m < n\} = W$ in $X$ which is a subspace of $X$.

If $(X, d)$ is a metric space, $\pi$ is a mapping in $X$ satisfies $m(\pi(\xi), \pi(ῄ)) \leq \theta\, m(\xi, ῄ)$ for $0 < \theta < 1$, $\xi, ῄ\epsilon X$ , $m$ is a metric, then $\pi$ is a contraction on $X$ and a contraction fixes a unique $s$ in $X$. That is, $\pi(s) = s$ for a unique $s$ in $X$. ...... 1.1

The set of linear operators defined over a linear space is denoted by $L(X)$. A one-one linear operator $\alpha\epsilon L(X)$ is invertible. Such linear operators can be called nonsingular.
If a nonsingular smooth linear operator $\alpha$ is invertible at a point $(s, t)$ in a linear space $X$, $\beta\, \epsilon L(X)$ with $\|\alpha - \beta\|\|\alpha^{-1}\| < 1$, then $\beta$ is also invertible at $(s, t)$. The set of invertible mappings on a metric space is $\partial$ which is an open set in $C'(X)$ which is the class of smooth linear operators. ...... 1.2

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The mapping $K: \alpha \to \alpha^{-1}$ is a continuous mapping on $\partial$.

If $\alpha \epsilon C'(X)$ is from an open set $\varphi$ and a subset of the linear space $X$, $\alpha'$ is invertible at $\xi \epsilon \varphi$, then $\alpha'$ is invertible in the neighbourhood of $\xi$ which is an open set $\varphi_1$as discussed as above. Further, the pre image of this open set is also an open set $\varphi_2 \subseteq \varphi$ such that the mapping $\beta: \varphi_2 \to \varphi_1$ satisfying $\beta(\alpha(\xi)) = \xi$ is also in $C'(X)$ stated as inverse function theorem. ...... 1.3

A linear mapping $\pi(s,t)$ in $C'(X)$ where $X$ is a linear space of dimension $2n$ with $\xi$ stands for the 1st $n$ components and ή for the remaining $n$ components of the vector ($s$, $t$) can be written as $\pi(s,t) = \pi_\xi s + \pi_{ή} t$ ...... 1.4

A linear mapping $\pi(s,t) \, \epsilon \, C'(X)$ with $(\xi, ή)$ a point in $\Gamma$ an open subset of $X$ a linear space of dimension $2n$ such that $\pi(\xi, ή) = \mathbf{0}$, $\pi'(\xi, ή) = M$ with $M = M_s\lambda + M_t\mu$ in which $M_s$ is invertible, then there exists open sets $\Psi$ in $\Gamma$ and φ in $X$ such that $\pi: \Psi \to \varphi$ is a bijection or nonsingular. Further, if $\rho: \varphi \to \Psi$ is the inverse function of $\pi$, then $\rho$ is $C'(X)$ satisfying $\rho(\pi(s,t)) = (s,t)$ for every $(s,t) \, \epsilon \, \Psi$ ...... 1.5

If $\pi(s,t)$ is a smooth function in an open set $\Gamma \subseteq X$ a finite dimensional linear space $X$ into a linear space $Y$ and $\pi'(s,t)$ is invertible for every ($s$, $t$) in $\Gamma$, then $\pi(\Psi)$ is an open set in $Y$ for every $\Psi$ an open set in $X$ which is the neighbourhood of ($s$, $t$). In particular, $\pi$ maps open sets onto open sets. So, $\pi$ is an open mapping. ...... 1.6

When a vector valued function $\alpha$ is smooth in an open set $\Gamma$ in a metric space $X$ into itself, and is invertible at one point ($s$, $t$) in $\Gamma$ where $\alpha(s,t) = \mathbf{0}$, and$\alpha'(s,t) = M$, then $M = M_\xi s + M_{ή} t$ such that $M_\xi$is invertible in the neighbourhood of ($s$, $t$). if this result is extended to the smooth linear mappings $\{\alpha\}$ defined from linear spaces of higher dimension to that of lower dimensions, then upto equal dimensions, invertibility of $\alpha$ is possible and there by the inverse function of $\alpha$ upto nonsingularity exists, and the remaining part of linear space $X$ is mapped to the **0** vector, satisfies $\alpha(\rho(s)) = s$ for every $s$ in $\Gamma$. Also, the non singular part of $\rho$ which is $\tau$ is the implicit function that satisfies

$$\tau'(t) = -M_\xi^{-1} M_{ή} \quad . \qquad \text{...... 1.7}$$

In the absence of continuity of the derivative of $\alpha$ in $\Gamma$, the invertibility of $\alpha$ upto equal dimensions will become impossible. Further, if the dimension of the linear space $X$ is double that [4] of the co-domain [1] linear space, then two such implicit functions are possible. These are mutually singular. This allows that the span of these functions is $X$. ...... 1.8

Two measures $\pi_{1\perp}$ and $\pi_2$ on a linear space $X$ are said to be mutually singular [2] if $\pi_2(\Psi_1) = \mathbf{0}$ and $\pi_1(\Psi_2) = \mathbf{0}$ for $\Psi_1$ and $\Psi_2$ are subspaces of $X$ such that $\Psi_1 \cap \Psi_2 = \{\mathbf{0}\}$ ...... 1.9

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## <u>Chapter 2</u>:

The comment in the article "An implicit function theorem" in the 'Journal of Optimization theory and Applications 'June 1980, is not justified. Perhaps, ***Sadatoshi Kumagai*** has not followed the text properly. He stated that the proof is not perfect and it can be given directly. I feel, It must be read through the perspective of contraction, continuously differentiability of a vector valued function or a multi variate function $\pi$ that leads to all the members ($s$, $t$) of the linear space $X$ in its neighbourhood are invertible and inverse function theorem. The theorem is meant for only those domains which is an open set containing the vectors where the given continuously differentiable function $\pi$ is invertible. So, to throw light onto the bizarre looks such as Kumagai, the word 'upto' is included in the present discussion. Upto the equal dimensions of the domain and co-domain, the invertibility prevails. As a whole, Kumagai's comment does not stand in front of the proof provided by Walter Rudin. Frankly i disagree with Kumagai's comment and uphold Rudin's explanation.

To throw light onto the Rudin's explanation and create insight, let me reiterate the statement in a more elaborate way and to extend this view point to more than one inverse functions, the following is one way.

"If $\pi$ is a smooth mapping from an open set $\Gamma$, a subset of a linear space $X$ of dimension $n$ + $m$, into a linear space $Y$ of dimension $n$, $(s,t)\epsilon\Gamma$ where $\pi'$ is invertible $\pi'(s,t) = M$ and to maintain the bijection or nonsingularity between the subsets that are neighbourhoods of $(s,t)$ and $\pi(s,t)$, the 1$^{st}$ $n$ – components(or any set of $n$ – components) of members $(s,t)$ in $X$ are mapped into the $n$ components of members of $Y$, the remaining $m$ – components of $(s,t)$are mapped to the zero vector in $Y$ satisfying the inverse part of $\pi$ as is the implicit function that is shown in (1.7). ...... 2.1

When $m$ = $n$, this view point is taken forwards as two equal splits of $X$ such that both can satisfy the one to one between the open sets of $X$ and $Y$ or the two invertible parts of $\pi$ onto $Y$ in view of (1.4), and thus, either $n$ – components of the members of $X$ form the null space with respect to $\pi$ leading to mutual singularity of the two implicit functions using the rank and nullity perspective of the linear mapping (transformation). ...... 2.2

## <u>Chapter 3</u>:

Definition: If $\pi$ is a continuous function from a linear space $X$ of dimension 2$n$ into a linear space of dimension $n$, $\pi = \pi_1 + \pi_2$ with $\pi_1$ maps the set of $n$ components of the members of $X$ and $\pi_2$ carries the remaining $n$ – components of the vectors of $X$ such that the domain of $\pi_1$ is the null space of $\pi_2$ and vice versa, then $\pi_1$ and $\pi_2$ are said to be mutually singular denoted by $\pi_1 \perp \pi_2$. ...... 3.1

Theorem: if $\pi_1, \pi_2 \epsilon C'(X,Y)$ from the open subsets

$\vartheta_1 = \{(s_{11}, s_{12}, \dots, s_{1n}, t_{11}, t_{12}, \dots, t_{1n}): s_{1i}, t_{1i} \,\epsilon\, F \;\forall\; 1 \le i \le n\}$ and ...... 3.2

$\vartheta_2 = \{(s_{21}, s_{22}, \dots, s_{2n}, t_{21}, t_{22}, \dots, t_{2n}): s_{2i}, t_{2i} \,\epsilon\, F \;\forall\; 1 \le i \le n\}$ ...... 3.3

are the disjoint subsets of a linear space $X$ of dimension 2$n$ into a linear space $Y$ of dimension $n$, $(s_1, t_1)$ and $(s_2, t_2)$ are points in $\vartheta_1$ and $\vartheta_2$ respectively such that

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$\pi_1(s_1, t_1) = \mathbf{0}$ and $\pi_2(s_2, t_2) = \mathbf{0}$ , ...... 3.4
$\pi_1(s_{2i}, t_{2i}) = \mathbf{0}$ and $\pi_2(s_{1i}, t_{1i}) = \mathbf{0}$ for $1 \le i \le n$ ...... 3.5

$\pi_1{}'(s_1, t_1) = M_1$ and $\pi_2{}'(s_2, t_2) = M_2$ ...... 3.6
with $M_{1s}$ and $M_{2t}$are invertible at these points, then there exists open sets $\Psi_1$ in $\vartheta_1$ , $\Psi_2$ in $\vartheta_2$ and $\varphi_{11}, \varphi_{22}$ in *Y* such that the linear functions $\tau_1\colon \varphi_{11} \to \Psi_1$ and $\tau_2\colon \varphi_{22} \to \Psi_2$ are the open mappings with the null space of $\pi_1$ as $\Psi_2$ and that of $\pi_2$ is $\Psi_1$ leading to $\pi_1 {}^\perp \pi_2$.

Proof: Define $\alpha(\xi, ή) = (\pi_1(\xi, ή), ή)$ & $\beta(\xi, ή) = (\xi, \pi_2(\xi, ή))$ for $(\xi, ή)$ in *K* ...... 3.7

Observe that $\alpha\colon \vartheta_1 \subseteq X \to X$ and $\beta\colon \vartheta_2 \subseteq X \to X$ are the mappings.

$\alpha(s_1 + \lambda, t_1 + \mu) = M_1(\lambda, \mu) + \theta_1(\lambda, \mu)$
$\beta(s_2 + \lambda, t_2 + \mu) = M_2(\lambda, \mu) + \theta_2(\lambda, \mu)$ where $\theta_1$ and $\theta_2$ are small positive quantities tend to 0.

$$\alpha(s_1 + \lambda, t_1 + \mu) - \alpha(s_1, t_1) = \pi_1\big((s_1 + \lambda, t_1 + \mu), \mu\big)$$
$$= (M_1(\lambda, \mu), \mu) + (\theta_1(\lambda, \mu), \mathbf{0})$$

$$\beta(s_2 + \lambda, t_2 + \mu) - \beta(s_2, t_2) = (\lambda, \pi_2\big((s_2 + \lambda, t_2 + \mu)\big)$$
$$= (\lambda, M_2(\lambda, \mu)) + (\mathbf{0}, \theta_2(\lambda, \mu))$$

$$\lim_{(\lambda,\mu)\to(\mathbf{0},\mathbf{0})} \frac{\alpha(s_1 + \lambda, t_1 + \mu) - \alpha(s_1, t_1)}{(\lambda, \mu)} = (M_1, \mathbf{0})$$
$$\lim_{(\lambda,\mu)\to(\mathbf{0},\mathbf{0})} \frac{\beta(s_2 + \lambda, t_2 + \mu) - \beta(s_2, t_2)}{(\lambda, \mu)} = (\mathbf{0}, M_2)$$

In view of (3.5), $\alpha'(s_1, t_1)$ is a linear operator that maps $(\lambda, \mu)\epsilon\vartheta_1$ to $(M_1(\lambda, \mu), \mu)\epsilon X$.

Similarly, $\beta'(s_2, t_2)$ is the linear operator *X* that maps $(\lambda, \mu)\epsilon\vartheta_2$ into $(\lambda, M_2(\lambda, \mu))\epsilon X$

What follows, $\alpha'$ and $\beta'$ are smooth in the neighbourhoods of $(s_1, t_1)$ and $(s_2, t_2)$ respectively. ...... 3.8

Since $(\lambda, \mu)$ is arbitrary in both $\vartheta_1$,$\vartheta_2$, and in view of (3.2), (3.3), (3.4), (3.6) and (3.7)allow $\alpha$ and $\beta$ satisfy (1.3).

In view of (3.4), $\mu = \mathbf{0}$ leads to $M_1(\lambda, \mathbf{0}) = \mathbf{0}$
$M_1 = M_{1\xi}\lambda + M_{1ή}\mathbf{0} = \mathbf{0},$ $M_{1\xi}$ is invertible and (1.4), allow $\lambda$= $\mathbf{0}$
What follows $M_1(\lambda, \mu) = \mathbf{0}$ implies $(\lambda, \mu) = \mathbf{0}$ ...... 3.9
$\alpha'$ is one - one from the 1st *n* components (or any set of *n* components) of $(\xi, ή)\epsilon\vartheta_1$ onto the open subset $(\mathbf{0}, ή)$ in *X* for ή in *Y*.

With this, applying (1.3) to $\alpha$, the neighbourhood of $(s_1, t_1)$ in $\vartheta_1$, an open set $\varphi_1$ and its pre – image $\Psi_1$ in *X* also an open set in view of (1.6).

For every$(\xi, ή)$ $\epsilon\Psi_1$, there corresponds $(\mathbf{0}, ή)\epsilon$ $\varphi_1 \subseteq X$ such that $\alpha(\xi, ή) =$ $(\pi_1(\xi, ή), ή)$
$= (\mathbf{0}, ή)$ ...... 3.10

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So, allowing $ή\epsilon\varphi_{11} \subseteq Y$, it follows $(\mathbf{0}, ή)\epsilon\varphi_1$ and $(\xi, ή)\ \epsilon\Psi_1$ that satisfies (3.10). depending on the openness of $\varphi_1$, it follows $\varphi_{11}$ is open.

$\pi_1(\xi, ή) = \mathbf{0} = \pi_1(\xi', ή)$ implies $\alpha(\xi, ή) = (\mathbf{0}, ή) = \alpha(\xi', ή)$ and $\alpha$ is non singular.
So, $(\xi, ή) = (\xi', ή)$ allows $\xi = \xi'$.
Consequently, for each $ή\epsilon\varphi_{11}$, there corresponds $\xi$ uniquely such that $\pi_1(\xi, ή) = \mathbf{0}$ for each $(\xi, ή)$ in $\Psi_1$.

For this$(\xi, ή)\ \epsilon\Psi_1$, the suitable implicit function $\tau_1: Y \to X$ defined by $\tau_1(ή) =$
$(\pi_1(\xi, ή), ή)$=$(\mathbf{0}$, ή) ...... 3.11

Observe that $(\tau_1(ή), ή) = \rho_1(\mathbf{0}, ή) = \zeta_1(ή)$ gives $\rho_1$ is smooth and so,
$\zeta_1{}'(ή)\mu = (\tau_1'(ή)\mu, \mu)$ for each $ή\epsilon\varphi_{11}$ and $\mu\ \epsilon\ Y$. ...... 3.12

See that $\pi_1(\zeta_1(ή)) = \mathbf{0}$ in $\varphi_{11}$ and so, (3.11) gives $\pi_1{}'(\zeta_1(ή))\zeta_1{}'(ή) = \mathbf{0}$
Since this property is true for every $ή\ \epsilon\varphi_{11}$ and $t_1\epsilon\varphi_{11}$, it follows $M_1\zeta_1{}'(t_1) = \mathbf{0}$

In view of (1.4) and (1.7), it follows $M_{1\xi}\tau_1'(t_1)\mu + M_{1ή}\mu = M_1\zeta_1{}'(t_1)\mu = \mathbf{0}$
Since $\mu$ in *Y* is arbitrary, it follows $M_{1\xi}\tau_1'(t_1) + M_{1ή} = \mathbf{0}$

From this, what follows $\tau_1'(t_1) = -{M_{1\xi}}^{-1}M_{1ή}$ ...... 3.13

Repeating the entire argument for $\beta(s_2, t_2)$, it follows $\tau_2'(s_2) = -{M_{2ή}}^{-1}M_{2\xi}$ ...... 3,14

Observe that, for every $(\lambda, \mu)$ in the neighbourhood $\Psi_2 \subseteq X$ of$(s_2, t_2)$, there corresponds $\lambda\ \epsilon\ \varphi_{22}$ an open set in *Y* such that $\left(\xi, \tau_2(\xi)\right) = \rho_2(\xi, \mathbf{0}) = \zeta_2(\xi)$ gives $\rho_2$ is smooth and so,
$\zeta_2{}'(\xi)\lambda = (\xi, \tau_2'(\xi)\lambda, \lambda)$ for each $\xi\epsilon\varphi_{22}$ and $\lambda\ \epsilon\ Y$.
$\varphi_{22}$ is the neighbourhood of $s_2\epsilon Y$, $(s_2, \mathbf{0})\epsilon X$ and $(s_2, t_2)\epsilon\Psi_2$ such that $\pi_2(s_2, t_2) = \mathbf{0}$ ...... 3.15

Applying (1.4), on both $\alpha$ & $\beta$, the open sets $\Psi_1$ & $\Psi_2$ in $X$ with
$\alpha(s_1, t_1)$ =$(\pi_1$ $(\mathbf{0}$,$t_1)$,$t_1) = (\mathbf{0}, t_1)\ \epsilon\ \varphi_1$ and $(t_{11}, t_{12}, \dots, t_{1n})\epsilon\ \varphi_{11}$ and
$\beta(s_2, t_2)$ = $(\pi_2(s_2, \mathbf{0})$,$s_2) = (s_2, \mathbf{0})\ \epsilon\ \varphi_2$, $(s_{21}, s_{22}, \dots, s_{2n})\ \epsilon\ \varphi_{22}$

In particular, $(\lambda, \mu)\epsilon\Psi_1$ implies $\beta(\lambda, \mu) = \left(\lambda, \pi_2(\lambda, \mu)\right) = (\lambda, \mathbf{0}) = (\mathbf{0}, \mu)\ \epsilon\ \varphi_1$ and $\lambda\epsilon\varphi_{22}$
$\lambda = \mathbf{0}, \mu = \mathbf{0}$

In view of (3.5), for every $(\lambda, \mu)\epsilon\Psi_1$, $\beta(\lambda, \mu) = \mathbf{0}$ implies $\pi_2(\lambda, \mu) = \mathbf{0}$
Similarly, $(\lambda, \mu)\epsilon\Psi_2$, $\alpha(\lambda, \mu) = \mathbf{0}$ implies $\pi_1(\lambda, \mu) = \mathbf{0}$ satisfying (1.9).
Therefore, $\pi_1 \perp \pi_2$.

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**References**:

[1] Srinivasarao, T., & Sujatha, L. Natural Absolutely Continuous and Mutually Singular Measures. IJERT, Vol.9 Issue 11, November – 2020

[2] **Mutually Singular Measurable Subspaces of a Linear Space"**, International Journal of Science & Engineering Development Research (www.ijsdr.org), ISSN:2455-2631, Vol.7, Issue 7, page no.610 - 611, July-2022, http://www.ijsdr.org/papers/IJSDR2207090.pdf

[3] https://link.springer.com/article/10.1007/BF00934117
https://scholar.google.com/scholar?hl=en&as_sdt=0%2C5&q=implicit+function+theorem&oq=implicit+f#d=gs_qabs&t=1723527662077&u=%23p%3DbMtE0iun6a0J

[4] https://onlinelibrary.wiley.com/doi/abs/10.1002/mana.19841150107

[5]
https://books.google.co.in/books?hl=en&lr=&id=w9jzCAAAQBAJ&oi=fnd&pg=PR5&dq=implicit+function+theorem+by+Walter+Rudin+Real+Analysis&ots=FszhhaadXL&sig=QlA9GehJgjxUvXviWYpIZEnYU2M&redir_esc=y#v=onepage&q&f=false