

Radio Number of Mycielski Graph of a Path

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Abstract

The radio labeling of a graph G is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ with the property that $|f(u) - f(v)| \geq 1 + diam(G) - d(u, v)$ for every pair of vertices $u, v \in V(G)$, where diam(G) is the diameter of G and d(u, v) is the distance between u and v in G. The radio number of G, denoted by rn(G), is the smallest integer k such that G admits a radio labeling. In this paper, we determine the radio number of Mycielski graph of a path.

Key Words: radio labeling, radio number, radio graceful graphs

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1 Introduction

All graphs in this paper are finite, simple, connected, and undirected. We use the standard terminology, the terms not defined here may be found in [1, 2]. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v, and is denoted by $d_G(u, v)$ or simply d(u, v). The maximum distance between any two vertices in G is called diameter of G and is denoted by diam(G).

Radio labeling is motivated by the channel assignment problem introduced by W. K. Hale et al [3] in 1980. The radio labeling of a graph is most useful in FM radio channel restrictions to overcome the effect of noise. The problem turns out to finding the minimum of maximum frequencies of all the radio stations considered under the network. The notion of radio labeling was introduced by G. Chartrand, D. Erwin, P. Zhang and F. Harary in [4]. Since the introduction of radio labeling, several authors have investigated the radio number of various networks [5, 6, 8, 9]. Recent work on radio labeling is found in [10, 11, 12, 13, 14].

In 1955, Jan Mycielski [7] introduced an interesting graph transformation which transforms a graph G into a new graph M(G), called the Mycielskian or the Mycielski graph of G. Using this construction, he created triangle-free graphs with large chromatic numbers. In 1998, David C. Fisher and et al [8] obtained the diameter of M(G).

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We determine the radio labeling of the Mycielski graph of a path in this paper. Some of the definitions and results on radio labeling are listed below for immediate reference.

Definition 1.0.1. A radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with v ∈ V (G) labeled by f(V), such that |f(u)−f(v)|+d(u, v) ≥ 1+diam(G) holds for all u, v ∈ V , u ≠ v. The radio number rn(f) of a radio labeling f of G is the maximum label assigned by f to a vertex of G. The radio number rn(G) of G is the min{rn(f)}, over all radio labelings f of G. A radio labeling f of G is a minimal radio labeling of G if rn(f) = rn(G).

Certainly, rn(G) ≥ n and f is injective. Further, if rn(G) = n, then graph G is radio graceful [9].

Definition 1.0.2. Let the n vertices of the given graph G be $v_1, v_2, \ldots, v_n$. The Mycielskian or Mycielski's Graph, denoted by M(G) contains G itself as a subgraph, together with n + 1 additional vertices: a vertex u_i corresponding to each vertex v_i of G and an extra vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge $v_i v_j$ of G, the Mycielski graph includes two edges $u_i v_j$ and $v_i u_j$

Thus, if G has n vertices and m edges, M(G) has 2n + 1 vertices and 3m + n edges.

1.1 Basic Results

Theorem 1.1.1 (Gary Chartrand, David Erwin, Ping Zhang [5]). Let Cn be the n-vertex cycle, n ≥ 3. Then

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n)+2, & if \quad n \equiv 0,2 \pmod{4} \\ \frac{n-1}{2}\phi(n)+1, & if \quad n \equiv 1,3 \pmod{4} \end{cases}$$

where $\phi(n) = \begin{cases} k+1, & if \quad n = 4k+1 \\ k+2, & if \quad n = k+r \quad for\ some\ r = 0,2,3 \end{cases}$

Theorem 1.1.2 (David c. Fisher, Patricia A. Mckenna, Elizabeth D. Boyer[8]).

For a graph G without isolated vertices, diam(M(G)) = min(max(2, diam(G)), 4).

Theorem 1.1.3. By Theorem 1.1.2 For any integer n ≥ 2, diam[M(P_n)] = $\begin{cases} 2, & if \quad n = 2,3 \\ 3, & if \quad n = 4 \\ 4, & if \quad n \geq 5 \end{cases}$

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2 Main Results obtained

Theorem 2.0.1. For any integer n ≥ 2, $rn[M(P_n)] = \begin{cases} 2n+1, & if \quad n = 2,3 \\ 4n-2, & if \quad n = 4 \\ 4n+5, & if \quad n = 5 \\ 4n+4, & if \quad n = 6 \\ 4n+2, & if \quad n \geq 7 \end{cases}$

Proof. Let $G = [M(P_n)]$. When n = 2, $M(P_n) \simeq C5$ result follows by Theorem 1.1.1. For n ≥ 3.
Let $v_1, v2, \dots, v_n, u_1, u_2, \dots, u_n, w$ be the vertices of G. Suppose f is a radio labeling of G and $x_1, x_2, \dots, x_m$, where m = 2n + 1 be the rearrangement of vertices of G such that
$f(x_i) < f(x_{i+1})$ then

$f(x_m) - f(x\,m-1) \geq 1 + d - d(x_m, x_{m-1})$

$f(x_{m-1}) - f(x_{m-2}) \geq 1 + d - d(x_{m-1}, x_{m-2})$

. . .

$f(x_2) - f(x1) \geq 1 + d - d(x_2, x_1)$

adding all these inequalities we get

$$f(x_m) - f(x_1) \geq (m-1)(1+d) - \sum_{i=1}^{m-1} d(x_i, x_{i-1})$$

Taking $f(x_1) = 1$

$$f(x_m) \geq 1 + (m-1)(1+d) - \sum_{i=1}^{m-1} d(x_i, x_{i-1})$$

This shows that $f(x_m)$ is minimum whenever $\sum_{i=1}^{m-1} d(x_i, x_{i-1})$ is maximum.

Therefore the problem reduces to the following linear integer programming problem

. That is max $z = \sum_{i=1}^{diam(G)} i\alpha_i$

subjected to, $\alpha_1 + \alpha_2 + \dots + \alpha_{diam(G)} = m - 1 = 2n$

where α_i is the number of pairs of edges of distance i and $\alpha'_i\, s \in Z^+$

(i) When n = 2, 3

We have by Theorem 1.1.3 $diam[M(P_n)] = 2$, for n = 2, 3.

max $z = \sum_{i=1}^{2} i\alpha_i$ subjected to, $\alpha_1 + \alpha_2 = m-1 = 2n$. By choosing, $\alpha_2 = m-1 = 2n$ and $\alpha_1 = 0$.

Therefore we get max $z = \sum_{i=1}^{2} i\alpha_i = 1 \times \alpha_1 + 2 \times \alpha_2 = 4n$. Thus $f(x_m) \geq 1 + (2n)(3) - 4n = 2n + 1$.

On the otherhand, the labeling shown in the following Figure 1 shows that $f(x_m) \leq 2n + 1$. Thus $rn[M(P_n)] = 2n + 1$, for n = 2, 3.

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Figure 1: Radio Number of $M(P_n)$ for the cases $n = 2, 3$.

(ii) When n = 4

We have by Theorem 1.1.3 diam $[M(P_n)] = 3$, for n = 4.

max $z = \sum_{i=1}^{3} i\alpha_i$ subjected to, $\alpha_1 + \alpha_2 + \alpha_3 = m-1 = 2n$. By choosing, $\alpha_3 = 3$, $\alpha_2 = 5$ and $\alpha_1 = 0$.

Therefore we get max $z = \sum_{i=1}^{3} i\alpha_i = 1 \times \alpha_1 + 2 \times \alpha_2 + 3 \times \alpha_3 = 19$. Thus $f(x_m) \geq 1 + (2n)(4) - 19$ $= 14 = 4n - 2$.

On the otherhand, the labeling shown in the following Figure 2 shows that $f(x_m) \leq 4n - 2$. Thus $rn[M(P_n)] = 4n - 2$, for n = 4.

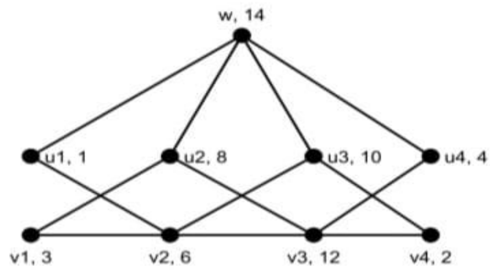

Figure 2: Radio Number of $M(P_n)$ for the case $n = 4$.

(iii) When n = 5

We have by Theorem 1.1.3 diam $[M(P_n)] = 4$, for n = 5.

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max z = $\sum_{i=1}^{4} i\alpha_i$ subjected to $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ = m − 1 = 2n. By choosing, α_4 = 0, α_3 = 6, α_2 = 4 and α_1 = 0. Therefore we get max z = $\sum_{i=1}^{4} i\alpha_i$ = 1 × α_1 + 2 × α_2 + 3 × α_3 + 4 × α_4 = 26. Thus f(x_m) ≥ 1 + (2n)(5) − 26 = 25 = 4n + 5.

On the otherhand, the labeling shown in the following Figure 3 shows that f(x_m) ≤ 25 = 4n + 5. Thus rn[M(P_n)] = 25 = 4n + 5, for n = 5.

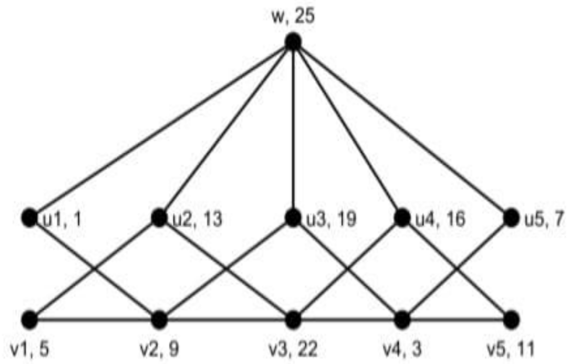

Figure 3: Radio Number of $M(P_n)$ for the case $n = 5$.

(iv) When n = 6

We have by Theorem 1.1.3 diam[M(P_n)] = 4, for n = 6.

max z = $\sum_{i=1}^{4} i\alpha_i$ subjected to $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ = m − 1 = 2n. By choosing, α_4 = 0, α_3 = 9, α_2 = 3 and

α_1 = 0. Therefore we get max z = $\sum_{i=1}^{4} i\alpha_i$ = 1 × α_1 + 2 × α_2 + 3 × α_3 + 4 × α_4 = 33. Thus

f(x_m) ≥ 1 + (2n)(5) − 33 = 28 = 4n + 4.

On the otherhand, the labeling shown in the following Figure 4 shows that f(x_m) ≤ 28 = 4n + 4. Thus rn[M(P_n)] = 28 = 4n + 4, for n = 6.

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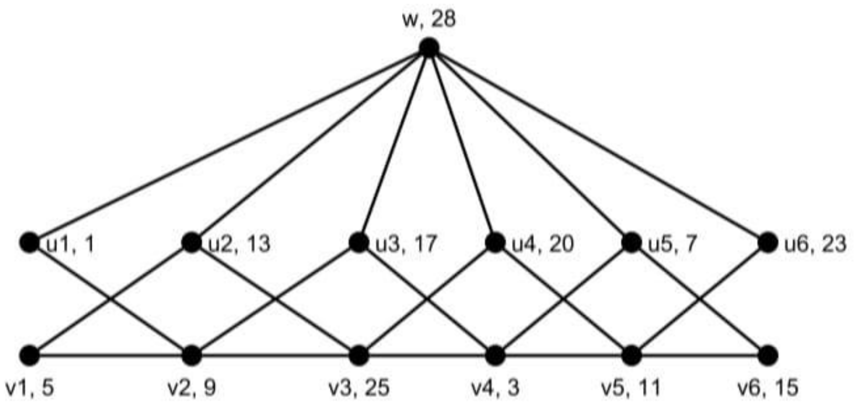

Figure 4: Radio Number of $M(P_n)$ for the case $n = 6$.

(v) When $n \geq 7$

We have by Theorem 1.1.3 diam[M(P_n)] = 4, for $n \geq 7$.

max $z = \sum_{i=1}^{4} i\alpha_i$ subjected to, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = m - 1 = 2n$. By choosing, $\alpha_4 = 0$, $\alpha_3 = 2n - 1$, $\alpha_2 = 1$ and $\alpha_1 = 0$. Therefore we get max $z = \sum_{i=1}^{4} i\alpha_i = 1\times\alpha_1 + 2\times\alpha_2 + 3\times\alpha_3 + 4\times\alpha_4 = 6n-1$. Thus $f(x_m) \geq 1+(2n)(5)-(6n-1) = 4n+2$. On the otherhand, we label the vertices by vertex ordering for G as, firstly we start from w to v_3 and later select $(2n-1)$ times P_3 path to cover all the vertices by taking alternating $v_i u_j$ -paths. The ordering of vertices is as follows $w - v_1 - u_{\lceil \frac{n}{2} \rceil} - v_n - u_{\lceil \frac{n}{2} \rceil - 1} - v_{n-1} -$

$u_{\lceil \frac{n}{2} \rceil - 2} - v_{n-2} - \ldots - u_2 - v_{\lceil \frac{n}{2} \rceil + 2} - u_1 - v_{\lceil \frac{n}{2} \rceil + 1} - u_n - v_{\lceil \frac{n}{2} \rceil} - u_{n-1} - v_{\lceil \frac{n}{2} \rceil - 1} - u_{n-2} - \ldots - v_3 - u_{\lceil \frac{n}{2} \rceil + 2} - v_2$ $- u_{\lceil \frac{n}{2} \rceil + 1}$.

By labeling the vertices in this order shows that $f(x_m) \leq 4n + 2$.

Thus rn[M(P_n)] = 4n + 2, for $n \geq 7$.

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