

Mutually Singular implicit functions

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Abstract A continuously differentiable (or **smooth** or in $C'(X_1)$) function π_1 from an open set Γ which is a subset of a linear space X_1 of dimension $2n$ into a linear space X_2 of dimension n has an inverse onto a subspace upto equal dimension. If (s, t) is in Γ where $\pi_1(s, t) = 0$, $\pi_1'(s, t) = M$ invertible at (s, t) , then π_1 is invertible in the neighbourhood of $(s, t) \in \Gamma$. Upto bijection, the invertibility of a multi variate function π_1 is possible and for this π_1 from X_1 is mapped onto X_2 so as to get invertible for the 1st n components of the $2n$ – tuple. In the present discussion, we focus on two multi variate functions π_1 and π_2 or linear mappings from X_1 into X_2 such that both are smooth in two open sets Γ_1 and Γ_2 subsets of X_1 taking the 1st n – components of $2n$ – tuples in X_1 onto X_2 while π_2 is from the open set Γ_2 of X_1 taking the 2nd set of n – components of $2n$ – tuples in X_1 such that π_1' is invertible at $(s_1, t_1) \in \Gamma_1$ and π_2' is invertible at $(s_2, t_2) \in \Gamma_2$. The sum of derivative of the implicit functions is a continuously differentiable function gives

$$\rho_1'(t) + \rho_2'(s) = - \{M_{1\xi}^{-1}M_{1\eta} + M_{2\eta}^{-1}M_{2\xi}\} \quad \text{..... 0.1}$$

with ρ_1 and ρ_2 are the implicit functions, $M_1 = M_{1\xi} + M_{1\eta}$ and $M_2 = M_{2\xi} + M_{2\eta}$ in which $M_{1\xi}$ and $M_{2\eta}$ are the invertible parts while $M_{1\eta}$ and $M_{2\xi}$ are singular parts.

Further, the null space of π_1 is the range space of π_2 and vice versa leading to π_1 and π_2 are mutually singular, denoted by $\pi_1 \perp \pi_2$ 0.2

Key words: linear space, multi variate function or a vector valued function, nonsingular mapping, mutually singular mapping, invertible mapping, null space $\vartheta(\pi)$, range space $\gamma(\pi)$, dimension of a linear space, span, rank and nullity of a linear transformation, contraction, upto invertibility, subset, open mapping.

Chapter 1:

Introduction: A linear space X of dimension n is of the form

$\{(s_1, s_2, \dots, s_n): s_i, 1 \leq i \leq n \in F\}$ where F is a field. Each n – tuple of X can be written as a linear combination of the n basis vectors of X . A proper subset of basis vectors can span $\{(t_1, t_2, \dots, t_m): t_i, 1 \leq i \leq m \in F, m < n\} = W$ in X which is a subspace of X .

If (X, d) is a metric space, π is a mapping in X satisfies $m(\pi(\xi), \pi(\eta)) \leq \theta m(\xi, \eta)$ for $0 < \theta < 1$, $\xi, \eta \in X$, m is a metric, then π is a contraction on X and a contraction fixes a unique s in X . That is, $\pi(s) = s$ for a unique s in X 1.1

The set of linear operators defined over a linear space is denoted by $L(X)$. A one-one linear operator $\alpha \in L(X)$ is invertible. Such linear operators can be called nonsingular.

If a nonsingular smooth linear operator α is invertible at a point (s, t) in a linear space X , $\beta \in L(X)$ with $\|\alpha - \beta\| \|\alpha^{-1}\| < 1$, then β is also invertible at (s, t) . The set of invertible mappings on a metric space is ∂ which is an open set in $C'(X)$ which is the class of smooth linear operators. 1.2

The mapping $K: \alpha \rightarrow \alpha^{-1}$ is a continuous mapping on ∂ .

If $\alpha \in C'(X)$ is from an open set φ and a subset of the linear space X , α' is invertible at $\xi \in \varphi$, then α' is invertible in the neighbourhood of ξ which is an open set φ_1 as discussed as above. Further, the pre image of this open set is also an open set $\varphi_2 \subseteq \varphi$ such that the mapping $\beta: \varphi_2 \rightarrow \varphi_1$ satisfying $\beta(\alpha(\xi)) = \xi$ is also in $C'(X)$ stated as inverse function theorem.

..... 1.3

A linear mapping $\pi(s, t)$ in $C'(X)$ where X is a linear space of dimension $2n$ with ξ stands for the 1st n components and η for the remaining n components of the vector (s, t) can be written as $\pi(s, t) = \pi_\xi s + \pi_\eta t$

..... 1.4

A linear mapping $\pi(s, t) \in C'(X)$ with (ξ, η) a point in Γ an open subset of X a linear space of dimension $2n$ such that $\pi(\xi, \eta) = \mathbf{0}$, $\pi'(\xi, \eta) = M$ with $M = M_s \lambda + M_t \mu$ in which M_s is invertible, then there exists open sets Ψ in Γ and φ in X such that $\pi: \Psi \rightarrow \varphi$ is a bijection or nonsingular. Further, if $\rho: \varphi \rightarrow \Psi$ is the inverse function of π , then ρ is $C'(X)$ satisfying $\rho(\pi(s, t)) = (s, t)$ for every $(s, t) \in \Psi$

..... 1.5

If $\pi(s, t)$ is a smooth function in an open set $\Gamma \subseteq X$ a finite dimensional linear space X into a linear space Y and $\pi'(s, t)$ is invertible for every (s, t) in Γ , then $\pi(\Psi)$ is an open set in Y for every Ψ an open set in X which is the neighbourhood of (s, t) . In particular, π maps open sets onto open sets. So, π is an open mapping.

..... 1.6

When a vector valued function α is smooth in an open set Γ in a metric space X into itself, and is invertible at one point (s, t) in Γ where $\alpha(s, t) = \mathbf{0}$, and $\alpha'(s, t) = M$, then $M = M_\xi s + M_\eta t$ such that M_ξ is invertible in the neighbourhood of (s, t) . If this result is extended to the smooth linear mappings $\{\alpha\}$ defined from linear spaces of higher dimension to that of lower dimensions, then upto equal dimensions, invertibility of α is possible and there by the inverse function of α upto nonsingularity exists, and the remaining part of linear space X is mapped to the $\mathbf{0}$ vector, satisfies $\alpha(\rho(s)) = s$ for every s in Γ . Also, the non singular part of ρ which is τ is the implicit function that satisfies

$$\tau'(t) = -M_\xi^{-1} M_\eta \quad . \quad \text{..... 1.7}$$

In the absence of continuity of the derivative of α in Γ , the invertibility of α upto equal dimensions will become impossible. Further, if the dimension of the linear space X is double that ^[4] of the co-domain ^[1] linear space, then two such implicit functions are possible. These are mutually singular. This allows that the span of these functions is X .

..... 1.8

Two measures $\pi_{1\perp}$ and π_2 on a linear space X are said to be mutually singular ^[2] if $\pi_2(\Psi_1) = \mathbf{0}$ and $\pi_1(\Psi_2) = \mathbf{0}$ for Ψ_1 and Ψ_2 are subspaces of X such that $\Psi_1 \cap \Psi_2 = \{\mathbf{0}\}$

..... 1.9

Chapter 2:

The comment in the article “An implicit function theorem” in the ‘Journal of Optimization theory and Applications’ June 1980, is not justified. Perhaps, **Sadatoshi Kumagai** has not followed the text properly. He stated that the proof is not perfect and it can be given directly. I feel, It must be read through the perspective of contraction, continuously differentiability of a vector valued function or a multi variate function π that leads to all the members (s, t) of the linear space X in its neighbourhood are invertible and inverse function theorem. The theorem is meant for only those domains which is an open set containing the vectors where the given continuously differentiable function π is invertible. So, to throw light onto the bizarre looks such as Kumagai, the word ‘upto’ is included in the present discussion. Upto the equal dimensions of the domain and co-domain, the invertibility prevails. As a whole, Kumagai’s comment does not stand in front of the proof provided by Walter Rudin. Frankly i disagree with Kumagai’s comment and uphold Rudin’s explanation.

To throw light onto the Rudin’s explanation and create insight, let me reiterate the statement in a more elaborate way and to extend this view point to more than one inverse functions, the following is one way.

“If π is a smooth mapping from an open set Γ , a subset of a linear space X of dimension $n + m$, into a linear space Y of dimension n , $(s, t) \in \Gamma$ where π' is invertible $\pi'(s, t) = M$ and to maintain the bijection or nonsingularity between the subsets that are neighbourhoods of (s, t) and $\pi(s, t)$, the 1st $n -$ components (or any set of $n -$ components) of members (s, t) in X are mapped into the n components of members of Y , the remaining $m -$ components of (s, t) are mapped to the zero vector in Y satisfying the inverse part of π as is the implicit function that is shown in (1.7). 2.1

When $m = n$, this view point is taken forwards as two equal splits of X such that both can satisfy the one to one between the open sets of X and Y or the two invertible parts of π onto Y in view of (1.4), and thus, either $n -$ components of the members of X form the null space with respect to π leading to mutual singularity of the two implicit functions using the rank and nullity perspective of the linear mapping (transformation). 2.2

Chapter 3:

Definition: If π is a continuous function from a linear space X of dimension $2n$ into a linear space of dimension n , $\pi = \pi_1 + \pi_2$ with π_1 maps the set of n components of the members of X and π_2 carries the remaining $n -$ components of the vectors of X such that the domain of π_1 is the null space of π_2 and vice versa, then π_1 and π_2 are said to be mutually singular denoted by $\pi_1 \perp \pi_2$ 3.1

Theorem: if $\pi_1, \pi_2 \in C'(X, Y)$ from the open subsets

$$\vartheta_1 = \{(s_{11}, s_{12}, \dots, s_{1n}, t_{11}, t_{12}, \dots, t_{1n}) : s_{1i}, t_{1i} \in F \forall 1 \leq i \leq n\} \text{ and} \quad \dots 3.2$$

$$\vartheta_2 = \{(s_{21}, s_{22}, \dots, s_{2n}, t_{21}, t_{22}, \dots, t_{2n}) : s_{2i}, t_{2i} \in F \forall 1 \leq i \leq n\} \quad \dots 3.3$$

are the disjoint subsets of a linear space X of dimension $2n$ into a linear space Y of dimension n , (s_1, t_1) and (s_2, t_2) are points in ϑ_1 and ϑ_2 respectively such that

$$\pi_1(s_1, t_1) = \mathbf{0} \text{ and } \pi_2(s_2, t_2) = \mathbf{0}, \quad \text{..... 3.4}$$

$$\pi_1(s_{2i}, t_{2i}) = \mathbf{0} \text{ and } \pi_2(s_{1i}, t_{1i}) = \mathbf{0} \text{ for } 1 \leq i \leq n \quad \text{..... 3.5}$$

$$\pi_1'(s_1, t_1) = M_1 \text{ and } \pi_2'(s_2, t_2) = M_2 \quad \text{..... 3.6}$$

with M_{1s} and M_{2t} are invertible at these points, then there exists open sets Ψ_1 in ϑ_1 , Ψ_2 in ϑ_2 and $\varphi_{11}, \varphi_{22}$ in Y such that the linear functions $\tau_1: \varphi_{11} \rightarrow \Psi_1$ and $\tau_2: \varphi_{22} \rightarrow \Psi_2$ are the open mappings with the null space of π_1 as Ψ_2 and that of π_2 is Ψ_1 leading to $\pi_1 \perp \pi_2$.

$$\text{Proof: Define } \alpha(\xi, \eta) = (\pi_1(\xi, \eta), \eta) \text{ \& } \beta(\xi, \eta) = (\xi, \pi_2(\xi, \eta)) \text{ for } (\xi, \eta) \text{ in } K \quad \text{..... 3.7}$$

Observe that $\alpha: \vartheta_1 \subseteq X \rightarrow X$ and $\beta: \vartheta_2 \subseteq X \rightarrow X$ are the mappings.

$$\alpha(s_1 + \lambda, t_1 + \mu) = M_1(\lambda, \mu) + \theta_1(\lambda, \mu)$$

$$\beta(s_2 + \lambda, t_2 + \mu) = M_2(\lambda, \mu) + \theta_2(\lambda, \mu) \text{ where } \theta_1 \text{ and } \theta_2 \text{ are small positive quantities tend to 0.}$$

$$\begin{aligned} \alpha(s_1 + \lambda, t_1 + \mu) - \alpha(s_1, t_1) &= \pi_1((s_1 + \lambda, t_1 + \mu), \mu) \\ &= (M_1(\lambda, \mu), \mu) + (\theta_1(\lambda, \mu), \mathbf{0}) \end{aligned}$$

$$\begin{aligned} \beta(s_2 + \lambda, t_2 + \mu) - \beta(s_2, t_2) &= (\lambda, \pi_2((s_2 + \lambda, t_2 + \mu))) \\ &= (\lambda, M_2(\lambda, \mu)) + (\mathbf{0}, \theta_2(\lambda, \mu)) \end{aligned}$$

$$\begin{aligned} \lim_{(\lambda, \mu) \rightarrow (\mathbf{0}, \mathbf{0})} \frac{\alpha(s_1 + \lambda, t_1 + \mu) - \alpha(s_1, t_1)}{(\lambda, \mu)} &= (M_1, \mathbf{0}) \\ \lim_{(\lambda, \mu) \rightarrow (\mathbf{0}, \mathbf{0})} \frac{\beta(s_2 + \lambda, t_2 + \mu) - \beta(s_2, t_2)}{(\lambda, \mu)} &= (\mathbf{0}, M_2) \end{aligned}$$

In view of (3.5), $\alpha'(s_1, t_1)$ is a linear operator that maps $(\lambda, \mu) \in \vartheta_1$ to $(M_1(\lambda, \mu), \mu) \in X$.

Similarly, $\beta'(s_2, t_2)$ is the linear operator X that maps $(\lambda, \mu) \in \vartheta_2$ into $(\lambda, M_2(\lambda, \mu)) \in X$

What follows, α' and β' are smooth in the neighbourhoods of (s_1, t_1) and (s_2, t_2) respectively. 3.8

Since (λ, μ) is arbitrary in both ϑ_1, ϑ_2 , and in view of (3.2), (3.3), (3.4), (3.6) and (3.7) allow α and β satisfy (1.3).

In view of (3.4), $\mu = \mathbf{0}$ leads to $M_1(\lambda, \mathbf{0}) = \mathbf{0}$

$$M_1 = M_{1\xi}\lambda + M_{1\eta}\mathbf{0} = \mathbf{0}, M_{1\xi} \text{ is invertible and (1.4), allow } \lambda = \mathbf{0}$$

What follows $M_1(\lambda, \mu) = \mathbf{0}$ implies $(\lambda, \mu) = \mathbf{0}$ 3.9

α' is one - one from the 1st n components (or any set of n components) of $(\xi, \eta) \in \vartheta_1$ onto the open subset $(\mathbf{0}, \eta)$ in X for η in Y .

With this, applying (1.3) to α , the neighbourhood of (s_1, t_1) in ϑ_1 , an open set φ_1 and its pre - image Ψ_1 in X also an open set in view of (1.6).

$$\begin{aligned} \text{For every } (\xi, \eta) \in \Psi_1, \text{ there corresponds } (\mathbf{0}, \eta) \in \varphi_1 \subseteq X \text{ such that } \alpha(\xi, \eta) &= (\pi_1(\xi, \eta), \eta) \\ &= (\mathbf{0}, \eta) \end{aligned} \quad \text{..... 3.10}$$

So, allowing $\eta \in \varphi_{11} \subseteq Y$, it follows $(\mathbf{0}, \eta) \in \varphi_1$ and $(\xi, \eta) \in \Psi_1$ that satisfies (3.10). depending on the openness of φ_1 , it follows φ_{11} is open.

$\pi_1(\xi, \eta) = \mathbf{0} = \pi_1(\xi', \eta)$ implies $\alpha(\xi, \eta) = (\mathbf{0}, \eta) = \alpha(\xi', \eta)$ and α is non singular.

So, $(\xi, \eta) = (\xi', \eta)$ allows $\xi = \xi'$.

Consequently, for each $\eta \in \varphi_{11}$, there corresponds ξ uniquely such that $\pi_1(\xi, \eta) = \mathbf{0}$ for each (ξ, η) in Ψ_1 .

For this $(\xi, \eta) \in \Psi_1$, the suitable implicit function $\tau_1: Y \rightarrow X$ defined by $\tau_1(\eta) = (\pi_1(\xi, \eta), \eta) = (\mathbf{0}, \eta)$ 3.11

Observe that $(\tau_1(\eta), \eta) = \rho_1(\mathbf{0}, \eta) = \zeta_1(\eta)$ gives ρ_1 is smooth and so, $\zeta_1'(\eta)\mu = (\tau_1'(\eta)\mu, \mu)$ for each $\eta \in \varphi_{11}$ and $\mu \in Y$ 3.12

See that $\pi_1(\zeta_1(\eta)) = \mathbf{0}$ in φ_{11} and so, (3.11) gives $\pi_1'(\zeta_1(\eta))\zeta_1'(\eta) = \mathbf{0}$
Since this property is true for every $\eta \in \varphi_{11}$ and $t_1 \in \varphi_{11}$, it follows $M_1\zeta_1'(t_1) = \mathbf{0}$

In view of (1.4) and (1.7), it follows $M_{1\xi}\tau_1'(t_1)\mu + M_{1\eta}\mu = M_1\zeta_1'(t_1)\mu = \mathbf{0}$
Since μ in Y is arbitrary, it follows $M_{1\xi}\tau_1'(t_1) + M_{1\eta} = \mathbf{0}$

From this, what follows $\tau_1'(t_1) = -M_{1\xi}^{-1}M_{1\eta}$ 3.13

Repeating the entire argument for $\beta(s_2, t_2)$, it follows $\tau_2'(s_2) = -M_{2\eta}^{-1}M_{2\xi}$ 3.14

Observe that, for every (λ, μ) in the neighbourhood $\Psi_2 \subseteq X$ of (s_2, t_2) , there corresponds $\lambda \in \varphi_{22}$ an open set in Y such that $(\xi, \tau_2(\xi)) = \rho_2(\xi, \mathbf{0}) = \zeta_2(\xi)$ gives ρ_2 is smooth and so, $\zeta_2'(\xi)\lambda = (\xi, \tau_2'(\xi)\lambda, \lambda)$ for each $\xi \in \varphi_{22}$ and $\lambda \in Y$.
 φ_{22} is the neighbourhood of $s_2 \in Y$, $(s_2, \mathbf{0}) \in X$ and $(s_2, t_2) \in \Psi_2$ such that $\pi_2(s_2, t_2) = \mathbf{0}$ 3.15

Applying (1.4), on both α & β , the open sets Ψ_1 & Ψ_2 in X with
 $\alpha(s_1, t_1) = (\pi_1(\mathbf{0}, t_1), t_1) = (\mathbf{0}, t_1) \in \varphi_1$ and $(t_{11}, t_{12}, \dots, t_{1n}) \in \varphi_{11}$ and
 $\beta(s_2, t_2) = (\pi_2(s_2, \mathbf{0}), s_2) = (s_2, \mathbf{0}) \in \varphi_2$, $(s_{21}, s_{22}, \dots, s_{2n}) \in \varphi_{22}$

In particular, $(\lambda, \mu) \in \Psi_1$ implies $\beta(\lambda, \mu) = (\lambda, \pi_2(\lambda, \mu)) = (\lambda, \mathbf{0}) = (\mathbf{0}, \mu) \in \varphi_1$ and $\lambda \in \varphi_{22}$
 $\lambda = \mathbf{0}, \mu = \mathbf{0}$

In view of (3.5), for every $(\lambda, \mu) \in \Psi_1$, $\beta(\lambda, \mu) = \mathbf{0}$ implies $\pi_2(\lambda, \mu) = \mathbf{0}$
Similarly, $(\lambda, \mu) \in \Psi_2$, $\alpha(\lambda, \mu) = \mathbf{0}$ implies $\pi_1(\lambda, \mu) = \mathbf{0}$ satisfying (1.9).
Therefore, $\pi_1 \perp \pi_2$.

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