Application of Gauss Divergence theorem for linear Partial Differential Equation in the frame work of Green's Function

Dr.B.Venkateswarlu, D.Vasu Babu, Dr.T.Koteswararao, T.Srinivasarao

Abstract: a feasible region in three dimensions is enclosed by a surface [6] $z = f(x, y)$ whose dynamic equation $x = x(t), y = y(t), z = f(x(t), y(t))$can be interpreted as the Gaussian surface found using the Gauss – Divergence theorem[4] that says "the outward drawn normal component multiplied with the tangent plane or the closed region as the projection of $z = f(x, y)$ or simply outward flux of the vector field κ upon the closed surface Σ is equal to the volume of the divergence of κ enclosed by Σ. Further, the flux using the arbitrary point that can be extended to the entire surface Σ which is equal to the circulation along the closed curve $\mathfrak{J}$ that encloses the projection $\mathfrak{R}$ of $z = f(x, y)$. Simply, a linear programming problem can be created from volume to the surface and then from the projection surface to the circulation.
On the other hand, for the numerical data having more data points, for sake of ease, the method of cubic spline interpolation is considered optimal. A polynomial of higher degree can be written as the product of linear and quadratic polynomials in a complex plane by fundamental theorem of algebra. So, a linear 2nd order partial differential equation can be seen as two 2nd order ordinary differential equations by fixing each argument of one variable as constant and solve the equation for the 2nd variable. So, to solve a 2nd order differential equation that has two implicit variables *t* and *s*, one can apply a Green's function and solve it. The action is suitable to both the explicit variables x and y, equating the solutions of both the differential equations at each case, we can get the generalization that represents the required integral surface.

Key words: 2nd order linear partial differential operator, Outward flux, volume integral, Gauss Divergence theorem, boundary conditions, numerical solutions, enclosing curve, normal, surface integral, linearly independent solutions

Introduction:

The construction of a partial differential equation $L(z) = f(x, y, z, \omega, \wp)$ from the two integral curve as the intersection of surfaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$ is
$L(z) = \chi_1(x, y)z_{xx} + \chi_2(x, y)z_{xy} + \chi_3(x, y)z_{yy}$,$\omega = z_x, \wp = z_y$ and this equation can be transformed into the three canonical forms [7] depending on the discriminant
$\chi_2{}^2 - 4\chi_1\chi_3 > 0, = 0, < 0$ respectively of $\chi_1\lambda^2 + \chi_2\lambda + \chi_3 = 0$
…… 0.1
In particular, the tangent line of intersection of the tangent planes to $\mathfrak{R}_1$ and $\mathfrak{R}_2$ having the point of contact along the integral curve has the directional cosines proportional to $\mathfrak{R}_1 \times \mathfrak{R}_2$. …… 0.2
When $\mathfrak{R}_1$ or $\mathfrak{R}_2$ is replace by the differential operator ∇, then the point of contact will be the point on the tangent surface to the integral surface of the intersecting surfaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$. So, the tangent element or the plane element formed as $\nabla \times \mathfrak{R}_1$ and $\nabla \times \mathfrak{R}_2$ …… 0.3
Under the application of the Gauss divergence theorem, results in the volume of the integral surfaces when the integral surface formed by (0.3) is a closed region $\mathfrak{R}$ enclosed by the surface Σ. In each case of the canonical form, the 2nd order linear partial differential equation can be

 https://doi.org/10.5281/zenodo.17230526

written as the 2^{nd} order ordinary differential equation by fixing either x or y in each of their respective arguments as in the case of numerical integration. …… 0.4

Now the 2^{nd} order linear ordinary differential equation has the internal variables s and t such that the cases $s< t$ and $s>t$ which is solved using the Green's function twice by considering y as constant once and x as constant in the 2^{nd} case. One additional parameter can be taken and solved to see that the Green's function suitable to the actual partial differential equation is one and the same in both the cases of y as constant and x as constant after finding the particular integral. …… 0.5

Through this, the arbitrary point on the surface $P(\xi,\eta)$ equation is identified and now the Gauss divergence theorem is applied to find the volume of the closed region enclosed by the surface through $P(\xi,\eta)$. …… 0.6

1. *Fitting green's function to solve homogeneous BVP*:

Let the solution of the linear partial differential equation is the closed surface or integral surface $z = f(x,y)$ is written for each case [7]of y, $c \le y \le d$, the independent variable x is a continuous variable in $a \le x \le b$. So, the cross section of the surface $z = f(x,y)$ is $y = y_1, z = f(x, y_1)$ which is a curve parallel to $XZ -$ plane in the Euclidean space $\mathbb{R}^3$. Taking the implicit variables $a \le t, s \le b$ for x for the verification of saddle points at some stage between a and b where conditions $t \le s$ & $t \ge s$ is used to include the [13] ***Green's function*** Γ_1. The particular case $y = y_1$ makes the linear partial differential equation $L\{(\mathfrak{J}_1 z_1')' + \lambda z_1\} + f_1(x) = 0$ where $z_1 = \frac{\partial z}{\partial x}$ with the boundary conditions $\alpha_{11} z(a) + \alpha_{12} z'(a) = 0$; $\alpha_{13} z(b) + \alpha_{14} z'(b) = 0$ …… 1.1

that are satisfied by the definition at the arbitrary point

$P(\xi,\eta), a \le \xi(t) \le b, c \le \eta(t) \le d, t = s$ is $\Gamma_1(t,s) = \begin{Bmatrix} \Gamma_{11}(t,s), L(\Gamma_{11}) = 0\,, t < s, \\ \Gamma_{12}(t,s), L(\Gamma_{12}) = 0\,, t > s \end{Bmatrix}$ …… 1.2

The purpose of the Green's function $\Gamma_1(t,s)$ in this discussion is finding $\zeta(t) = f(\xi(t),\eta(t))$. …… 1.3

For this, it is created that there is a jump discontinuity at $\zeta(\xi(t),\eta(t))$ for the continuous partial derivative of $z = \zeta(t) = \Gamma_1(t,s)$ and to remove the discontinuity, the boundary conditions will be used as follows.

When $y = y_1$, the given linear partial differential equation has already become the ordinary linear differential equation that has the implicit variable t compared with s which seem to be (1.1) given in the explanation with the boundary conditions (1.2) and (1.3). While (1.1) is an ordinary 2^{nd} order linear homogeneous boundary value problem, let its independent solutions be $x_1(t)$ and $x_2(t)$ such that $\Gamma_1(t,s) = \begin{Bmatrix} \beta_1 x_1(t), t \le s \\ \beta_2 x_2(t), t \ge s \end{Bmatrix}$ …… 1.4

β_1, β_2 are constants chosen such that

$\beta_2 x_2(t) - \beta_1 x_1(t) = 0$ at t = s, and $\beta_2 x_2'(s) - \beta_1 x_1'(s) = \frac{-1}{\mathfrak{J}_1(s)}$ …… 1.5

Since $L(\Gamma_1) = 0$ for all $a \le x \le b$ where $t < s$ and $t > s$ holds and further at $t = s$, (1.4) and (1.5) hold.

Keeping this in view, homogeneous (1.1) will become $0 = (\mathfrak{J}_1 x_1')' x_2 - (\mathfrak{J}_1 x_2')' x_1$
$= \mathfrak{J}_1 W(x_1, x_2)$ and $\mathfrak{J}_1 \neq 0$

https://doi.org/10.5281/zenodo.17230526

From this, $x_1'(s)x_2(s) - x_2'(s)x_1(s) = \frac{\mathcal{C}}{\mathfrak{J}_1(s)}$ for some constant $\mathcal{C}$ …… 1.6

(1.5) and (1.6) together give $\beta_1 = \frac{-x_2(s)}{\mathcal{C}}$ and $\beta_2 = \frac{-x_1(s)}{\mathcal{C}}$ …… 1.7

Now, the required Green's function (1.4) that satisfies the homogenous ordinary linear differential equation for $y = y_1$ is $\Gamma_1(t,s) = \begin{Bmatrix} \frac{-x_2(s)x_1(t)}{\mathcal{C}}, t \leq s \\ \frac{-x_1(s)x_2(t)}{\mathcal{C}}, t \geq s \end{Bmatrix}$ …… 1.8

2. *Extending the Green's function to solve nonhomogeneous BVP*:

Let us extend the solution of the homogeneous ordinary linear differential equation to the non – homogeneous equation (1.1), suppose ${\Gamma_1}^1(t,s) = \chi_1 \Gamma_1(t,s)$ is the solution of (1.1). …… 2.1

Then, $\left\{\left(\mathfrak{J}_1({\Gamma_1}^1)'\right)' + \lambda {\Gamma_1}^1\right\} = -f(x, y_1)$ and y_1 is a constant.

Multiplying the Green's function on both sides and integrating between the boundary conditions, It gives the particular integral ${\Gamma_1}^1(t,s) = \int_a^b f(x, y_1)\, \Gamma_1(t,s) ds$ …… 2.2

Therefore, the general solution of the non homogeneous linear partial differential equation (1.1) is $z(x(t), y_1) = \Gamma_1(t,s) + {\Gamma_1}^1(t,s)$ …… 2.3

For an arbitrary $y = y_i, c \leq y_j(t) \leq d$, (2.3) becomes $z\left(x(t), y_j\right) = \Gamma_1(t,s) + {\Gamma_1}^1(t,s)$ …… 2.4

By mimicking the above argument by fixing x as $x = x_i, a \leq x_i(t) \leq d,$ the linear ordinary non homogeneous differential equation $\left\{\left(\mathfrak{J}_2({\Gamma_2}^1)'\right)' + \lambda {\Gamma_2}^1\right\} = -f(x_i, y)$ …… 2.5

is $z\left(x_i, y(t)\right) = \Gamma_2(t,s) + {\Gamma_2}^1(t,s)$ …… 2.6

While $z = f(x,y)$ is the closed surface or region Σ enclosed by a curve Ψ and so, the Green's theorem results in

$\{(\mathfrak{J}_1 {z_1}')' + \lambda z_1\} + f_1(x) = 0$ where $z_1 = \frac{\partial z}{\partial x}$ …… 3.1

The boundary conditions $\alpha_{11} z(a) + \alpha_{12} z'(a) = 0$ …… 3.2

$\alpha_{13} z(b) + \alpha_{14} z'(b) = 0$ …… 3.3

The function $\Gamma\ (t,s): [a,b] \times [a,b] \to [h,k]$ where $h \leq z \leq k$ is satisfied by $\Gamma(t,s) = \begin{Bmatrix} (2.4) \\ (2.6) \end{Bmatrix}$ …… 3.4

This is even though the scalar valued function, through the use of the variables *t* and *s*, treat this as a vector valued function defined at the arbitrary point or particle at $z(x,y) \leq \zeta(\xi,\eta)$ in the mass of the volume and the surface is [10] $z(x,y) = \zeta(\xi,\eta)$.

$z(t_1$, s)= $\int_a^b \Gamma_1(t_1,s) f(x, y_i) ds$ where t_1 and y_i are continuous variables. …… 3.5

$z(t_2$, s)= $\int_c^d \Gamma_2(t_2,s) f(x_j, y) ds$ where t_2 and x_j are continuous variables. …… 3.6

In view of [6], (3.5), (1.4) result in

$[z]_P = \frac{1}{2}\{[\Gamma_1] + [\Gamma_2]\} - \int_a^b \Gamma(\mathfrak{J}_2 dx - \mathfrak{J}_1 dy) - \frac{1}{2}\int \beta_i x_i(t)(\Gamma_{1y} dy - \Gamma_{2x} dx)$

 https://doi.org/10.5281/zenodo.17230526

$$-\frac{1}{2}\int \gamma_j y_j(t)\left(\Gamma_{2x}dx - \Gamma_{1y}dy\right) + \iint (\Gamma f dxdy)_\Sigma \quad \ldots\ldots 3.7$$

Similarly, (3.6) and (1.4) result in the same surface of z at P.

Isolating $[z]_P$ from this by applying Gauss – divergence theorem[14], it gives

$$[z]_P = \frac{1}{2}\{[\Gamma_1] + [\Gamma_2]\} - \int_a^b \Gamma(\mathfrak{J}_2 dx - \mathfrak{J}_1 dy) - \frac{1}{2}\int \beta_i x_i(t)(\Gamma_{1y}dy - \Gamma_{2x}dx)$$
$$-\frac{1}{2}\int \gamma_j y_j(t)\left(\Gamma_{2x}dx - \Gamma_{1y}dy\right) + \iiint (\nabla \cdot \nabla \kappa dV)_{\mathfrak{R}}$$

References:

1. https://books.google.com/books?hl=en&lr=&id=Ott1EAAAQBAJ&oi=fnd&pg=PP1&dq=Gauss+divergence+theorem+and+partial+differential+equations&ots=cVJAuE0VsL&sig=tE7CGmjUXuUnrD0uCG2hufkp9S8

2. Text book on "Applications of Green's Functions in Science And Engineering", Mikhael D.Greenberg, Department of Mechanical and Aerospace Engineering, University of Delaware, Dover Publication, Inc. Mineola, New York, 2015, ISBN: 978 – 0 – 486 – 79796 – 0

3. https://link.springer.com/content/pdf/10.1007/978-3-319-02099-0.pdf

4. arXiv:1406.4349v2[Math.AP] 9 Mar2015, A generalization of Gauss Divergence Theorem

5. Partial Differential Equations, 2nd Edition, Lawrence C.Evans, American Mathematical Society, 2010, ISBN: 978 – 0 – 8218 – 4974 – 3

6. Elements of Partial Differential Equations, Ian N Sneddon, Dover Publications, Inc., Mineola, New York, ISBN: 978 – 0 – 486- 49876 – 8

7. Text book of Ordinary Differential Equations, Tata McGraHill, 3rd edition, SGDeo, V Raghavendra, V.Lakshmikantham, ISBN: 978 – 93 – 392 – 1938 – 7

8. *Katz, Victor (2009). "Chapter 22: Vector Analysis". A History of Mathematics: An Introduction. Addison-Wesley. pp. 808–9. ISBN 978-0-321-38700-4.*

9. https://link.springer.com/content/pdf/10.1007/978-3-319-02099-0.pdf

10. https://link.springer.com/10.1007/s10559-024-00669-4?fromPaywallRec=true

11. https://books.google.com/books?hl=en&lr=&id=puM9DwAAQBAJ&oi=fnd&pg=PR9&dq=

divergence+theorem+and+partial+differential+equation&ots=1le4N1IZGb&sig=7NLTKHKlsZp--QVT8fbnMLEEd1o

12. Taylor, Michael E. (2011). "Partial Differential Equations I". *Applied Mathematical Sciences*. Vol. 115. New York, NY: Springer New York. pp. 178–179. doi:10.1007/978-1-4419-7055-8. ISBN 978-1-4419-7054-1. ISSN 0066-5452.

13. **Volume 32-Issue 8, 2025(*Journal of Dalian University of Technology*)** https://doi.org/10.5281/Zenodo.16925132

14. *M. R. Spiegel; S. Lipschutz; D. Spellman (2009). Vector Analysis. Schaum's Outlines (2nd ed.). USA: McGraw Hill. ISBN 978-0-07-161545-7.*

Dr. B. Venkateswarlu; Asst. Prof., Dept. of Math., P.G.Center, Kavali, Vikram Simhapuri University drbvvsuck@gmail.com

D.Vasu Babu; Asst.Professor;Dept. of Math.Krishna University, Dvb.amaths.aca@kru.ac.in

Dr.Thamma Koteswararao