A Study on Discrete Time Pre-Emptive Retrial Markov Chain with Change of Vacation Times

S. PAVAI MADHESWARI[1], S.A. JOSEPHINE[2*], P. SUGANTHI[3]

[1]*Professor, Department of Mathematics, R.M.K. Engineering College, Kavaraipettai - 601206, Tamil Nadu, India.*
[2*]*Assistant Professor, Department of Mathematics, Loyola-ICAM College of Engineering and Technology, Chennai-34, Tamil Nadu, India.*
[3]*Assistant Professor, R.M.K. Engineering College,Kavaraipettai - 601206, Tamil Nadu, India,*
∗ Corresponding author; S.A. JOSEPHINE

ABSTRACT

We consider a discrete time retrial queue with customer displacement, Bernoulli vacation and possibility of changing the remaining vacation period. If an arriving customer finds the server to be free, his service commences immediately. On the other hand, if the server is found to be busy, the customer either displaces the customer at service with probability θ to start his service or decides to join the pool of blocked customers with complementary probability $\bar{\theta}$ to retry for service after a random period of time. Upon completion of a service, the server chooses to go on vacation with probability p or continues to serve the next customer with complementary probability $\bar{p}$. During a vacation period, changes in remaining vacation times are permitted based on requirements. A detailed study of the system is performed. Using the probability generating function approach the probability generating functions of the orbit size and system size are derived. The effect of the parameters on the performance measures are analytically derived and numerically validated. Stochastic decomposition property has been established for the system size.

KEYWORDS**:** *Discrete Time Queues, Customer Retrial, Displacement, Vacation, Changes in Vacation Times.*

INTRODUCTION

Retrial queues are characterised by the feature that upon arrival if the server is not available to serve him immediately(may be busy serving other customers or on vacation), the arriving customer will join the orbit and try for his service after some random amount of time. There arise several situations where the server alternates between an active mode and an inactive mode. The server may be assigned some other work as soon as the busy period ends, the server may not be waiting for customers after completing the service of the last customers in the system. This type of queueing situation where the server may not be available for the next customer who arrives to an empty system is referred to as a queueing system with vacations. Retrial vacation queues are found to have applications in telephone switching systems, telecommunication networks, computer networks, manufacturing systems etc., as retrying for service is a general phenomenon in all these situations. The research in the areas of retrial queues and queues with vacation have focused primarily on the continuous time models. However, some real time situations such as wireless sensor networks and digital communication systems which operate on a discrete time basis where the events can happen only at regularly spaced time epochs made the discrete time retrial queues more suitable for analysis. Further, due to it's broader applicability in the performance analysis of production and inventory systems and telecommunication networks including the Broadband Services Digital Network(B-ISDN) which operates on Asynchronous Transfer Mode(ATM) and in related fields, the discrete time vacation queueing systems have attracted both queueing theorists and Electronics and Communication Engineers rapidly.

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Focusing on a sensor node in a WSN, the node itself can be viewed as a server and the messages as customers. The random times at which the node senses a data would be the arrival process and the transmission times would be the service process. The time the sensor node is being idle (sleep mode) can be taken as the vacation period. The sleep mode can be extended if there are no data packets to be transmitted which will save the battery power. Further, when one message is being processed in a sensor node there may be a new message arriving to that node which may be more important than the one being processed. In such case the message under processing will be made to wait and the arrived message is transmitted immediately. On the other hand the arriving message enters the buffer(orbit) to try again later. In many such applications the sensor nodes are deployed in distant, unattended and hostile environment with large quantities and the size of the sensor node is made to be very small. It is usually difficult to recharge or replace their batteries as most of the sensor nodes are equipped with non-rechargeable batteries that have limited lifetime. We attempt to use the discrete time retrial queue to model such a WSN.

REVIEW OF RELATED LITERATURE

Discrete time retrial queues are investigated by many researchers in the recent years. Frank, L. (2017) and Tsaklidis, G. & Vasiliadis, G. (2011) have explored discrete time queues. Nobel(2016) presented a survey on retrial queueing models in discrete time. An overview of queueing systems with vacations can be found in Upadhyaya(2016). Discrete time queues with vacation have been studied by Alfa(1995, 1998, 2003) and he has given some decomposition results (Alfa(2014))for a class of vacation queues. Atencia(2016) investigated a discrete time queueing system with vacations wherein he has considered the possibility of altering the vacation duration. In the recent years, there has been an increasing interest in the study of queues with service interruptions and priorities. Queueing systems with service interruptions are discussed by Krishnamoorthy et. al.,(2012), Krishnamoorthy et. al.,(2013) and Atencia(2015). Wu et. al., (2013) have investigated a discrete time retrial queue with preferred and impatient customers. Imen Bouazzi et. al., (2017) and Ke et. al.,(2015) have experimentally shown that the energy consumption in the WSNs can be reduced and important data can be transmitted with less packet delay.

In this paper, we analyse a discrete time retrial queueing system with customer displacement and change of vacation which may be applied for energy conservation and transmitting the important messages with less delay in wireless sensor networks. The messages sensored by sensor nodes are of various levels of importance. A right judgement of the priority of packets according to the importance/urgency of data may reduce the delay of transmission of the important/urgent information. That is, the real time emergency messages should be delivered to the sink node with the shortest possible end to end delay. We assume displacement of customer by which the delivery order of the data packets is changed based on their importance. The assumption that the vacation period can be shortened or elongated ensures the time period the transmitter is in the sleep mode may be shortened if any messages have arrived before the end of the vacation period (sleep period) or extended if no further messages have arrived at the end of the sleep period. It is shown by numerical illustrations that this assumption reduces the energy consumption thus extending the life time of the WSNs.

The rest of the paper is arranged as follows. Section 2 gives a brief discussion of the model under study. In section 3, we discuss the Markov chain of the system and exhibit the probability

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generating functions of the system in the idle state, busy state and vacation state. The PGF of the orbit size and system size have been derived. Some of the marginal generating functions have been presented and performance measures are derived. Section 4 enumerates some of the performance measures of the system. In Section 5, the stochastic decomposition of the system size is established. Section 6 gives the numerical results which are analogous to the analytical derivations followed by conclusion in Section 7.

THE MODEL

We consider a discrete time retrial queue where the time axis is partitioned into small subintervals of fixed length called slots. All queueing activities like arrival, departure, retrial, beginning/end of vacation etc., are assumed to occur only at the slot boundaries. Since in discrete time systems many events may occur concurrently, we need to clearly specify the order of occurence of the activities. An early arrival policy is considered in this study where the departures and end of vacations occur in sequence in (m^{-}, m) and primary arrivals, retrials, beginning of vacations occur in (m, m^{+}) in sequence. Since service always starts and ends at the slot boundaries, service times will be integral multiples of the slot lengths.

Primary arrivals are described by means of a geometric process with parameter $a > 0$, where a is the probability that an arrival occurs in a slot, with a maximum of one arrival per slot. Upon arrival if the server is free, the service of the customer is started immediately. On the contrary, if the server is busy the primary arrival either displaces the customer at service with probability θ to start his service or decides to join the pool of blocked customers with complementary probability $\bar{\theta}$ to retry for service after a random period of time. After a service completion, the server decides with probability p to go on vacation and with probability $\bar{p}(= 1 - p)$ continues to provide service to the next customer. When there are no customers in the system, the server does not go on a vacation immediately. He chooses to go on a vacation with probability p or decides to wait for the next arrival with probability $\bar{p}$. Upon arrival, if the server is found to be on vacation, the primary arrival joins the orbit to retry at a later instant. During a vacation period, changes in the number of remaining vacation slots is permitted with probability ν. It is reasonably assumed that in the slot in which vacation begins there is no change in the vacation slots.

The retrial times, service times and vacation times are assumed to follow general distributions $\{r_i\}_{i=0}^{\infty}$, $\{s_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ with probability generating functions $R(x) = \sum_{i=0}^{\infty} r_i x^i$, $S(x) = \sum_{i=1}^{\infty} s_i x^i$ and $V(x) = \sum_{i=1}^{\infty} v_i x^i$ respectively. It is assumed that service times, vacation times and retrial times are independent and identically distributed. In an attempt to avoid trivial cases, we assume that, $0 < p < 1, 0 < \theta < 1, 0 < \nu < 1$.

STEADY STATE DISTRIBUTION

At time m^{+}, the instant immediately after time slot m, the system can be described by the process $Y_m = (C_m, \xi_{0,m}, \xi_{1,m}, \xi_{2,m}, N_m)$, where C_m represents the server state 0, 1 or 2 according to the server being free, busy or on vacation respectively. N_m denotes the number of customers in the orbit. If $C_m = 0$ and $N_m > 0$ then $\xi_{0,m}$ denotes the remaining retrial time. If $C_m = 1$, $\xi_{1,m}$ corresponds to the remaining service time of the customer being served. If $C_m = 2$ and $N_m > 0$ then $\xi_{2,m}$ gives the remaining vacation slots. It can be proved that $\{Y_m : m \in N\}$ is the Markov chain of the queueing system , whose state space is:

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$$\{\{(0,0)\} \cup \{(0,i,k): i \geq 1, k > 0\} \cup \{(1,i,k): i \geq 1, k \geq 0\} \cup \{(2,i,k): i \geq 1, k \geq 0\}\}.$$

In order to obtain the stationary distribution of the process, we define the stationary probabilities of the Markov chain Y_m as follows:

$$\pi_{0,0} = \lim_{m\to\infty} P[C_m = 0; N_m = 0],$$
$$\pi_{j,i,k} = \lim_{m\to\infty} P[C_m = j; \xi_{j,m} = i; N_m = k], j = 1,2, i \geq 1, k \geq 0; j = 0, k > 0.$$

The Kolmogorov equations for the stationary distribution are given as follows:

$$\pi_{0,0} = \overline{a}\pi_{0,0} + \overline{ap}\pi_{1,1,0} + \overline{a}\pi_{2,1,0}, \quad (1)$$

$$\pi_{0,i,k} = \overline{a}\pi_{0,i+1,k} + \overline{ap}r_i\pi_{1,1,k} + \overline{a}r_i\pi_{2,1,k}, \quad i \geq 1, k \geq 1, \quad (2)$$

$$\pi_{1,i,k} = a\delta_{0,k}s_i\pi_{0,0} + (1-\delta_{0,k})\sum_{j=1}^{\infty} as_i\pi_{0,j,k} + \overline{a}s_i\pi_{0,1,k+1} + as_i\overline{p}\pi_{1,1,k} +$$
$$+\overline{a}r_0\overline{p}s_i\pi_{1,1,k+1} + as_i\pi_{2,1,k} + (1-\delta_{0,k})a\overline{\theta}\pi_{1,i+1,k-1} + \overline{a}\pi_{1,i+1,k} + \overline{a}r_0s_i\pi_{2,1,k+1} +$$
$$+(1-\delta_{0,k})\sum_{j=2}^{\infty} a\theta s_i\pi_{1,j,k-1}, \quad i \geq 1, k \geq 0 \quad (3)$$

$$\pi_{2,i,k} = apv_i(1-\delta_{0,k})\pi_{1,1,k-1} + \bar{a}pv_i\pi_{1,1,k} + (1-\delta_{0,k})a\bar{v}\pi_{2,i+1,k-1}$$
$$+\bar{a}\bar{v}\pi_{2,i+1,k} + (1-\delta_{0,k})\sum_{j=2}^{\infty} avv_i\pi_{2,j,k-1} + \sum_{j=2}^{\infty} \bar{a}vv_i\pi_{2,j,k}, \quad i \geq 1, k \geq 0, \quad (4)$$

where $\overline{a} = 1-a; \overline{v} = 1-v;\ \overline{p} = 1-p;\ \overline{\theta} = 1-\theta$ and $\delta_{i,j}$ denotes Kronecker delta.

The normalisation condition is

$$\pi_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \pi_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{1,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{2,i,k} = 1.$$

To solve equations (1) to (4), we define the following generating functions:

$$\phi_0(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \pi_{0,i,k}x^iz^k, \quad \phi_1(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{1,i,k}x^iz^k.$$

and $\phi_2(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{2,i,k}x^iz^k$.

Also the auxiliary generating functions are defined as:

$\phi_{0,i}(z) = \sum_{k=1}^{\infty} \pi_{0,i,k}z^k, \quad i \geq 1,\ \phi_{1,i}(z) = \sum_{k=0}^{\infty} \pi_{1,i,k}z^k, \quad i \geq 1,$

and $\phi_{2,i}(z) = \sum_{k=0}^{\infty} \pi_{2,i,k}z^k, \quad i \geq 1$.

In the process of deriving the above auxiliary generating functions, multiplying equations (2) to (4) by z^k and summing over all values of k, we get

$$\phi_{0,i}(z) = \overline{a}\left[\phi_{0,i+1}(z) + \bar{p}r_i\phi_{1,1}(z) + r_i\phi_{2,1}(z)\right] - ar_i\pi_{0,0}, \quad i \geq 1, \quad (5)$$

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$$\phi_{1,i}(z) = \frac{\overline{a}s_i}{z}\phi_{0,1}(z) + [(\frac{\bar{a}r_0+az}{z})\overline{p} - a\theta z]s_i\phi_{1,1}(z) + [\bar{a} + a\bar{\theta}z]\phi_{1,i+1}(z) + as_i\phi_0(1,z)$$
$$+az\theta s_i\phi_1(1,z) + [\frac{\bar{a}r_0+az}{z}]s_i\phi_{2,1}(z) + a(\frac{z-r_0}{z})s_i\pi_{0,0}, \quad i \geq 1, \tag{6}$$

$$\phi_{2,i}(z) = [\bar{a} + az][pv_i\phi_{1,1}(z) + \bar{v}\phi_{2,i+1}(z) + \nu v_i\phi_2(1,z) - \nu v_i\phi_{2,1}(z)], \quad i \geq 1. \tag{7}$$

To derive the generating functions from the above auxiliary generating functions, multiplying (5) to (7) by x^i and summing over all values of i,

$$\frac{(x-\overline{a})}{x}\phi_0(x,z) = [R(x) - r_0][\overline{ap}\phi_{1,1}(z) + \overline{a}\phi_{2,1}(z) - a\pi_{0,0}] - \overline{a}\phi_{0,1}(z), \tag{8}$$

$$\frac{(x-\beta(z))}{x}\phi_1(x,z) = [[(\frac{\bar{a}r_0+az}{z})\overline{p} - a\theta z]S(x) - \beta(z)]\phi_{1,1}(z) + a(\frac{z-r_0}{z})S(x)\pi_{0,0}$$
$$+aS(x)\phi_0(1,z) + \frac{\bar{a}}{z}S(x)\phi_{0,1}(z) + a\theta zS(x)\phi_1(1,z) + (\frac{\bar{a}r_0+az}{z})S(x)\phi_{2,1}(z), \tag{9}$$

$$\frac{(x-(\bar{a}+az)\bar{v})}{x}\phi_2(x,z) = (\bar{a} + az)\{pV(x)\phi_{1,1}(z) + \nu V(x)\phi_2(1,z) - [\bar{v} + \nu V(x)]\phi_{2,1}(z)\}, \tag{10}$$

where $\beta(z) = \bar{a} + a\bar{\theta}z$.

Substituting $x = 1$ in (8) and (9) respectively, we obtain the unknown functions $\phi_0(1,z)$&$\phi_1(1,z)$ which when used in (9), we get

$$\frac{x-\beta(z)}{x}\phi_1(x,z) = \{(\frac{[z+\bar{a}r_0(1-z)](1-\bar{\theta}z)\bar{p}-\theta z^2}{z(1-z)})S(x) - \beta(z)\}\phi_{1,1}(z)$$
$$-\frac{a(1-\bar{\theta}z)r_0}{z}S(x)\pi_{0,0} + \frac{\bar{a}}{z}(1-\bar{\theta}z)S(x)\phi_{0,1}(z) \tag{11}$$
$$. \quad +\frac{(1-\bar{\theta}z)[z+\bar{a}r_0(1-z)]}{z(1-z)}S(x)\phi_{2,1}(z).$$

Substituting $x = 1$ in (10) to find the unknown constant $\phi_2(1,z)$ and replacing it again in (10), we obtain

$$a(1-z)\left[\frac{x-(\bar{a}+az)\bar{v}}{x}\right]\phi_2(x,z) =$$
$$= (\bar{a} + az)\{pV(x)[1 - (\bar{a} + az)\bar{v}]\phi_{1,1}(z)$$
$$- [\nu V(x) + a(1-z)\bar{v}]\phi_{2,1}(z)\}. \tag{12}$$

To evaluate the auxiliary generating functions we proceed as follows:

Substitute $x = \bar{a}$ in (8)

$$a[R(\bar{a}) - r_0]\pi_{0,0} = [R(\bar{a}) - r_0]\left[\bar{a}\bar{p}\phi_{1,1}(z) + \bar{a}\phi_{2,1}(z)\right] - \bar{a}\phi_{0,1}(z). \tag{13}$$

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Replace $x=\beta(z)$ in (11)

$$\frac{ar_0(1-\bar{\theta}z)S(\beta(z))}{z}\pi_{0,0}= \{(\frac{[z+\bar{a}r_0(1-z)](1-\bar{\theta}z)\bar{p}-\theta z^2}{z(1-z)})S(\beta(z))-\beta(z)\}\phi_{1,1}(z) + \frac{\bar{a}}{z}(1-\bar{\theta}z)S(\beta(z))\phi_{0,1}(z)+\frac{(1-\bar{\theta}z)[z+\bar{a}r_0(1-z)]}{z(1-z)}S(\beta(z))\phi_{2,1}(z). \quad (14)$$

Substitute $x=(\bar{a}+az)\bar{v}$ in (12)

$$[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\phi_{2,1}(z)=pV((\bar{a}+az)\bar{v})[1-(\bar{a}+az)\bar{v}]\phi_{1,1}(z) \quad (15)$$

Solving equations (13), (14) and (15), the auxiliary generating functions are obtained as follows:

$$\phi_{0,1}(z)=\frac{az[R(\bar{a})-r_0]\{(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]-\{(1-\bar{\theta}z)A(z)-\theta z^2[\nu V((\bar{a}+az)\bar{v}) +a(1-z)\bar{v}]\}S(\beta(z))\}}{\bar{a}\gamma(z)}\pi_{0,0} \quad (16)$$

$$\phi_{1,1}(z)=\frac{a(1-z)(1-\bar{\theta}z)R(\bar{a})S(\beta(z))[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]}{\gamma(z)}\pi_{0,0} \quad (17)$$

$$\phi_{2,1}(z)=\frac{ap(1-z)(1-\bar{\theta}z)R(\bar{a})[1-(\bar{a}+az)\bar{v}]S(\beta(z))V((\bar{a}+az)\bar{v})}{\gamma(z)}\pi_{0,0} \quad (18)$$

where,

$$\gamma(z)=\{\{[z+\bar{a}(1-z)R(\bar{a})](1-\bar{\theta}z)A(z)-\theta z^2[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}S(\beta(z)) -z(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}$$

$$A(z)=\bar{p}[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]+pV((\bar{a}+az)\bar{v})[1-(\bar{a}+az)\bar{(\nu)}]$$

Substituting (16) to (18) in (8), (11) and (12) respectively, we obtain the generating functions:

$$\phi_0(x,z)= [\frac{R(x)-R(\bar{a})}{(x-\bar{a})\gamma(z)}]\times azx\{(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}] +\{\theta z[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]-(1-\bar{\theta}z)A(z)\}S(\beta(z))\}\pi_{0,0,} \quad (19)$$

$$\phi_1(x,z)=[\frac{S(x)-S(\beta(z))}{x-\beta(z)}]\times\frac{ax(1-z)(1-\bar{\theta}z)\beta(z)R(\bar{a})[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]}{\gamma(z)}\pi_{0,0,} \quad (20)$$

$$\phi_2(x,z)=[\frac{V(x)-V((\bar{a}+az)\bar{v})}{x-(\bar{a}+az)}]\times\frac{axp\bar{v}(1-z)(\bar{a}+az)[1-(\bar{a}+az)\bar{v}](1-\bar{\theta}z)R(\bar{a})S(\beta(z))}{\gamma(z)}\pi_{0,0.} \quad (21)$$

Substituting $x=1$ and $z=1$ in (19), (20), (21) and using normalisation condition,

$\pi_{0,0}+\phi_0(1,1)+\phi_1(1,1)+\phi_2(1,1)=1$, we get $\pi_{0,0}$:

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$$\pi_{0,0} = \frac{-\gamma'(1)}{\theta\nu R(\bar{a})V(\bar{\nu})S(\bar{a}+a\bar{\theta})} \tag{22}$$

where, $\gamma'(1) = \{\nu(\bar{a}+a\bar{\theta})V(\bar{\nu}) - [\bar{a}\theta\nu R(\bar{a})V(\bar{\nu}) - ap\theta\bar{\nu} + ap\theta\bar{\nu}V(\bar{\nu}) + \nu V(\bar{\nu})]S(\bar{a}+a\bar{\theta})\}$.

We suppose that the condition $-\gamma'(1) > 0$ is fulfilled in the rest of the paper.

The above results are summarised in the following theorem.

Theorem 1 *The generating functions of the stationary distribution of the Markov chain* $\{Y_m : m \in N\}$ are given by:

$$\phi_0(x,z) = [\frac{R(x)-R(\bar{a})}{(x-\bar{a})}] \times \times \frac{\{axz\{(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}] + \{\theta z[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}] - (1-\bar{\theta}z)A(z)\}S(\beta(z))\}\}}{\gamma(z)}\pi_{0,0}.$$

$$\phi_1(x,z) = [\frac{S(x)-S(\beta(z))}{x-\beta(z)}] \times \frac{ax(1-z)(1-\bar{\theta}z)\beta(z)R(\bar{a})[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}]}{\gamma(z)}\pi_{0,0}$$

$$\phi_2(x,z) = [\frac{V(x)-V((\bar{a}+az)\bar{\nu})}{x-(\bar{a}+az)}] \times \frac{axp\bar{\nu}(1-z)(\bar{a}+az)[1-(\bar{a}+az)\bar{\nu}](1-\bar{\theta}z)R(\bar{a})S(\beta(z))}{\gamma(z)}\pi_{0,0}$$

where:

$$\gamma(z) = \{\{[z+\bar{a}(1-z)R(\bar{a})](1-\bar{\theta}z)A(z) - \theta z^2[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}]\}S(\beta(z)) - z(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}]\},$$

$$A(z) = \bar{p}[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}] + pV((\bar{a}+az)\bar{\nu})[1-(\bar{a}+az)\bar{(\nu)}],$$

$$\pi_{0,0} = \frac{-\gamma'(1)}{\theta\nu R(\bar{a})V(\bar{\nu})S(\bar{a}+a\bar{\theta})}$$

and $\gamma'(1) = \{\nu(\bar{a}+a\bar{\theta})V(\bar{\nu}) - [\bar{a}\theta\nu R(\bar{a})V(\bar{\nu}) - ap\theta\bar{\nu} + ap\theta\bar{\nu}V(\bar{\nu}) + \nu V(\bar{\nu})]S(\bar{a}+a\bar{\theta})\}$.

Corollary 1 *The marginal generating function of the number of customers in the orbit when the server is idle is given by*

$$\pi_{0,0} + \phi_0(1,z) = \frac{R(\bar{a})\{\{(1-\bar{\theta}z)A(z)(\bar{a}+az) - \theta z^2[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}]\}S(\beta(z)) - z(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{\nu}) + a(1-z)\bar{\nu}]\}}{\gamma(z)}\pi_{0,0}. \tag{23}$$

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Corollary 2 *The marginal generating function of the number of customers in the orbit when the server is busy is given by*

$$\phi_1(1,z) = \frac{[1-S(\beta(z))](1-z)\beta(z)R(\bar{a})[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]}{\gamma(z)}\pi_{0,0}. \quad (24)$$

Corollary 3 *The marginal generating function of the number of customers in the orbit when the server is on vacation is given by*

$$\phi_2(1,z) = \frac{ap\bar{v}(1-z)(\bar{a}+az)(1-\bar{\theta}z)[1-V((\bar{a}+az)\bar{v})]R(\bar{a})S(\beta(z))}{\gamma(z)}\pi_{0,0}. \quad (25)$$

Corollary 4 *The probability generating function of the orbit size N is given by*

$$N(z) = \pi_{0,0} + \phi_0(1,z) + \phi_1(1,z) + \phi_2(1,z),$$

$$N(z) = \frac{\begin{array}{l}\{R(\bar{a})\{S(\beta(z))\{(\bar{a}+az)(1-\bar{\theta}z)[A(z)+ap\bar{v}(1-z)[1-V((\bar{a}+az)\bar{v})]]-[\nu V((\bar{a}+az)\bar{v})\\ +a(1-z)\bar{v}][\theta z^2+(1-z)\beta(z)]\}+(1-z)^2\beta(z)[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}\}\end{array}}{\gamma(z)}\pi_{0,0}. \quad (26)$$

Corollary 5 *The probability generating function of the system size* L_s *is given by*

$$L(z) = \pi_{0,0} + \phi_0(1,z) + z\phi_1(1,z) + \phi_2(1,z),$$

$$L(z) = \frac{\begin{array}{l}R(\bar{a})S(\beta(z))\{(\bar{a}+az)(1-\bar{\theta}z)\{A(z)+ap\bar{v}(1-z)[1-V((\bar{a}+az)\bar{v})]\}\\ -z[\theta z+(1-z)\beta(z)][\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}\end{array}}{\gamma(z)}\pi_{0,0.} \quad (27)$$

PERFORMANCE MEASURES

System performance measurement depends on the predefined system goals and the measures for it's evaluation. Hence, analysis of queueing systems is crucial for optimal management of a system and desigining of cost effective congestion control for the same. Some performance measures of the system obtained at the stationary state are summarised below.

- The probability that the system is free is given by

$$\pi_{0,0} = \frac{S(\bar{a}+a\bar{\theta})\{V(\bar{v})[ap\theta\bar{v}+\nu+\bar{a}\theta\nu R(\bar{a})]-ap\theta\bar{v}\}-(\bar{a}+a\bar{\theta})\nu V(\bar{v})}{\theta\nu R(\bar{a})V(\bar{v})S(\bar{a}+a\bar{\theta})}.$$

- The probability that the system is occupied is given by

$$\phi_0(1,1) + \phi_1(1,1) + \phi_2(1,1) = \frac{\{(\bar{a}+a\bar{\theta})\nu V(\bar{v})-\{V(\bar{v})[\nu+ap\theta\bar{v}-a\theta\nu R(\bar{a})]-ap\theta\bar{v}\}S(\bar{a}+a\bar{\theta})\}}{\theta\nu R(\bar{a})V(\bar{v})S(\bar{a}+a\bar{\theta})}.$$

- The server is idle with probability

$$\pi_{0,0}+\phi_0(1,1)=\frac{R(a)\{S(\bar{a}+a\bar{\theta})\{[\bar{a}\theta\nu+ap\theta\bar{v}+\nu]V(\bar{v})-ap\theta\bar{v}\}-(\bar{a}+a\bar{\theta})\nu V(\bar{v})\}}{\theta\nu R(\bar{a})V(\bar{v})S(\bar{a}+a\bar{\theta})}.$$

- The server is busy with probability

$$\phi_1(1,1)=\frac{(\bar{a}+a\bar{\theta})\nu R(\bar{a})V(\bar{v})[1-S(\bar{a}+a\bar{\theta})]}{\theta\nu R(\bar{a})V(\bar{v})S(\bar{a}+a\bar{\theta})}.$$

- The server is on vacation with probability

$$\phi_2(1,1)=\frac{ap\bar{v}\theta R(\bar{a})S(\bar{a}+a\bar{\theta})[1-V(\bar{v})]}{\theta\nu R(\bar{a})V(\bar{v})S(\bar{a}+a\bar{\theta})}.$$

- The mean system sizeL_s is derived by differentiating (27) with respect to z and substituting $z=1$ as

$$L_s=L'(1)$$

$$=\frac{1}{2}\{\frac{\omega''(1)}{\omega'(1)}-\frac{\gamma''(1)}{\gamma'(1)}\}$$

where,

$$\omega(z)=\quad R(\bar{a})S(\beta(z))\{(\bar{a}+az)(1-\bar{\theta}z)\{A(z)+ap\bar{v}(1-z)[1-V((\bar{a}+az)\bar{v})]\} \\ -z[\theta z+(1-z)\beta(z)][\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}$$

and

$$\gamma(z)=\{\{[z+\bar{a}(1-z)R(\bar{a})](1-\bar{\theta}z)A(z)-\theta z^2[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}S(\beta(z)) \\ -z(1-z)\beta(z)[\nu V((\bar{a}+az)\bar{v})+a(1-z)\bar{v}]\}.$$

STOCHASTIC DECOMPOSITION

The property deals with decomposing the system size into two random variables, where the first one corresponds to the system size of the standard queue without vacations and the second is the size of the system corresponding to the additional customers due to the vacation of the server.

We note that the probability generating function corresponding to the system size can be exhibited as,

$$L(z)=[L(z)]_{r_0=1}\times\frac{\pi_{0,0}+\phi_0(1,z)}{\pi_{0,0}+\phi_0(1,1)}. \quad (28)$$

The decomposition property for the proposed model under study is presented in the following theorem.

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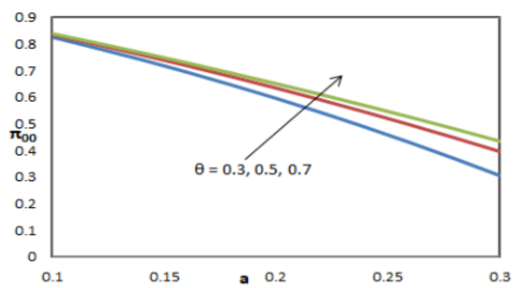
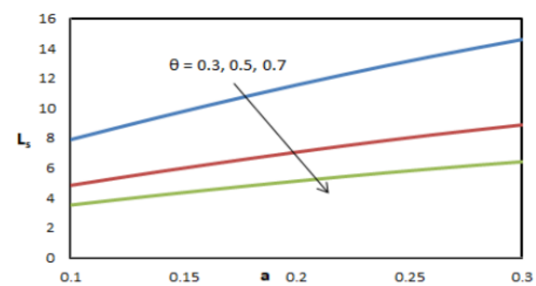
Theorem 2 *The system size* $E(L)$ *is decomposed as the addition of two independent random variables* L_1 *and* L_2*, i.e.,* $E(L) = L_1 + L_2$ *where* L_1 *is the system size in the* $Geo/G/1$ *system without vacation and* L_2 *is the additional size of the system developed as the server is on vacation.*

NUMERICAL RESULTS

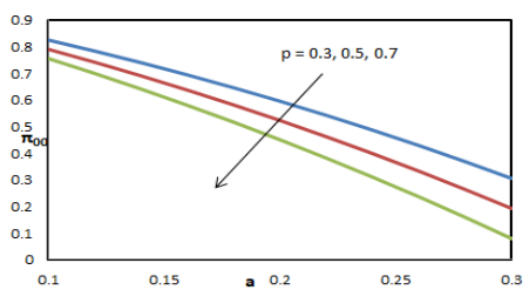
In this section the numerical validation of the analytical derivations are presented. The impact of some system parameters like arrival and displacement are on the probability of the system being empty, $\pi_{0,0}$ and the average system size, $E(L)$ are exhibited. Parameter values satisfying the stability condition are chosen for study. It is assumed that the retrial time, service time and vacation times follow geometric distribution with PGFs $R(x) = \frac{\bar{r}x}{1-rx}$, $S(x) = \frac{\bar{\mu}x}{1-\mu x}$ and $V(x) = \frac{\bar{\nu}x}{1-\nu x}$ where $r = 0.3$, $\mu = 0.4$ and $\nu = 0.4$ respectively.



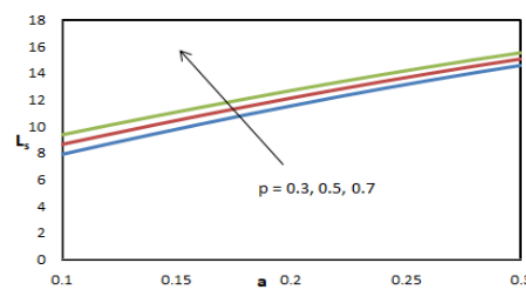
(a) $\pi_{0,0}$ versus a for $\theta = 0.3, 0.5, 0.7$



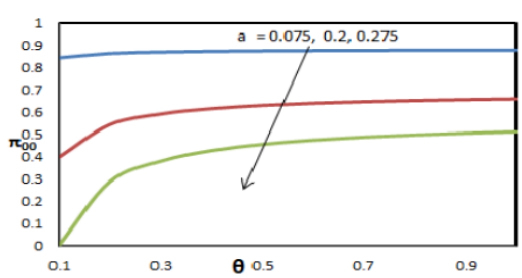
(b) $E(L)$ versus a for $\theta = 0.3, 0.5, 0.7$



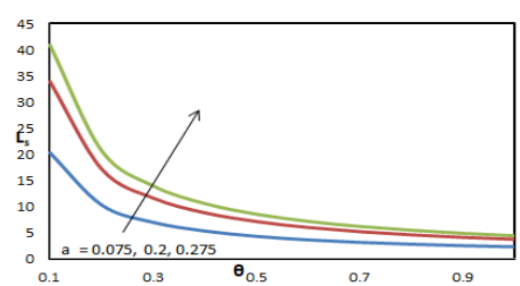
(c) $\pi_{0,0}$ versus a for $p = 0.3, 0.5, 0.7$



(d) $E(L)$ versus a for $p = 0.3, 0.5, 0.7$



(e) $\pi_{0,0}$ versus θ for $a = 0.075, 0.2, 0.275$



(f) $E(L)$ versus θ for $a = 0.075, 0.2, 0.275$

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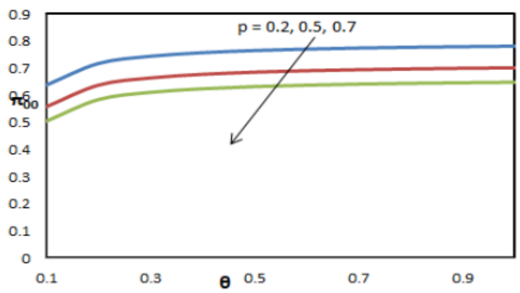
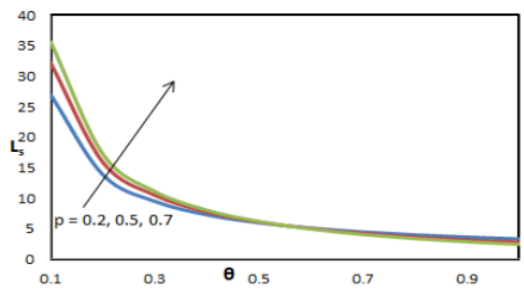


(g) $\pi_{0,0}$ versus θ for $p = 0.2, 0.5, 0.7$



(h) $E(L)$ versus θ for $p = 0.2, 0.5, 0.7$

In Figures (a) and (b) $\pi_{0,0}$ and $E(L)$ against the arrival probability a for different displacement probabilities are plotted. The curve of $\pi_{0,0}$ is seen to steadily decrease while on the contrary $E(L)$ increases for increasing values of a, which supports our intuition. In Figures (c) and (d) $\pi_{0,0}$ and $E(L)$ are drawn against the arrival probability a for different vacation probabilities and the trend of both the curves are as expected and the curve for $\pi_{0,0}$ is observed to be high for lower values of vacation probability p. The reverse trend is noted in the graph for $E(L)$, for increasing values of p, the values of $E(L)$ are found to be higher as anticipated.

In Figures (e) and (f) the system empty probability $\pi_{0,0}$ and the expected number of customers $E(L)$ are marked against the displacement probability θ, for various values of arrival probability a. The curve of $\pi_{0,0}$ increases for increasing values of the displacement probability while that of the curve for $E(L)$ gradually decreases for increasing values of the θ. The Figures (g) and (h) are drawn for $\pi_{0,0}$ and $E(L)$ across the displacement probability for various values of vacation probabilities p. As anticipated, the curve of $\pi_{0,0}$ is observed to have a steady increase for increasing values of the displacement probability θ. On the contrary, the graph for $E(L)$ decreases for increasing values of θ. The graph of $\pi_{0,0}$ is found to be higher for lower values of p.

CONCLUSION

In this paper, a discrete time retrial queueing system with displacement and change of vacation times is analysed. Introduction of change of vacation times in a retrial queue enables the customer to retry for service if the server is found to be busy and also the server is given an opportunity to change the number of slots for vacation based on the current requirement. The probability generating function technique has been applied and the generating functions of the Markov chain in the steady state are obtained. Important generating functions have been derived. Some performance measures of the system are obtained. Numerical results are presented to illustrate the effect of some system parameters on the performance of the system.

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